

Robust Stability of Linear Time-Delay Systems with Unknown Parameters and Delays

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ABSTRACT This paper investigates the robust stability of a class of linear time-delay systems subject to unknown parameters and delays. Since the exact values of these quantities are often unavailable in practice, an analytical framework based on the distribution of characteristic roots is developed. Through theoretical derivation of the characteristic equation for the perturbed system, an upper bound on its unstable eigenvalues is established. Subsequently, by employing the argument principle, sufficient conditions for robust stability are derived for both first-order and high-order time-delay systems. These conditions are easy to verify with low computational cost and are convenient for direct engineering application.

INDEX TERMS Argument principle, Linear time-delay systems, Robust stability, Unknown delays, Uncertain parameters

I. INTRODUCTION

CONSIDER the actual linear time-delay system subject to unknown parameters and delays, described by

$$\dot{x}(t) = (A_0 + \Delta A_0)x(t) + \sum_{i=1}^m (A_i + \Delta A_i)x(t - \tilde{\tau}_i). \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state variable, the constant matrices $A_0, A_i \in \mathbb{R}^{n \times n}$ ($i = 1, 2, \dots, m$) are known, the matrices $\Delta A_0, \Delta A_i \in \mathbb{R}^{n \times n}$ ($i = 1, 2, \dots, m$) are assumed unknown, and $\tilde{\tau}_i > 0$ ($i = 1, 2, \dots, m$) are unknown delays. System (1) is viewed as a perturbation of the following nominal system:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^m A_i x(t - \tau_i), \quad (2)$$

where τ_i ($i = 1, 2, \dots, m$) are known delays that differ from $\tilde{\tau}_i$. Let τ_i denote the nominal delays and $\tilde{\tau}_i$ the perturbed delays.

When modeling the system, the exact structures of the matrices $\Delta A_0, \Delta A_i \in \mathbb{R}^{n \times n}$ ($i = 1, 2, \dots, m$) and the time delays $\tilde{\tau}_i$ ($i = 1, 2, \dots, m$) are not known a priori. Nevertheless, the following boundedness information is assumed to be available. The norm of the uncertain matrices is bounded by a prescribed constant α :

$$\|\Delta A_0\| + \sum_{i=1}^m \|\Delta A_i\| \leq \alpha, \quad (3)$$

and the delay mismatches are bounded by prescribed constants α_i :

$$|\tilde{\tau}_i - \tau_i| \leq \alpha_i, \quad i = 1, 2, \dots, m. \quad (4)$$

We further consider linear high-order time-delay systems subject to unknown parameters and delays:

$$x^{(n)}(t) + \sum_{l=1}^n \left[(A_l + \Delta A_l)x^{(n-l)}(t) + \sum_{i=1}^m (B_{l,i} + \Delta B_{l,i})x^{(n-l)}(t - \tilde{\tau}_i) \right] = 0. \quad (5)$$

where $A_l, B_{l,i} \in \mathbb{R}^{n \times n}$ ($l = 1, 2, \dots, n, i = 1, 2, \dots, m$) are known constant matrices, $\Delta A_l, \Delta B_{l,i} \in \mathbb{R}^{n \times n}$ ($l = 1, 2, \dots, n, i = 1, 2, \dots, m$) are unknown matrices, and $\tilde{\tau}_i > 0$ ($i = 1, 2, \dots, m$) are unknown delays. System (5) is the perturbed form of the nominal system:

$$x^{(n)}(t) + \sum_{l=1}^n \left[A_l x^{(n-l)}(t) + \sum_{i=1}^m B_{l,i} x^{(n-l)}(t - \tau_i) \right] = 0. \quad (6)$$

where τ_i ($i = 1, 2, \dots, m$) are known nominal delays that differ from $\tilde{\tau}_i$, and condition (4) is also assumed to hold for the high-order systems. Moreover, the perturbed matrices $\Delta A_l, \Delta B_{l,i} \in \mathbb{R}^{n \times n}$ ($l = 1, 2, \dots, n, i = 1, 2, \dots, m$) are assumed to have known norm bounds, satisfying

$$\sum_{l=1}^n \left[\|\Delta A_l\| + \sum_{i=1}^m \|\Delta B_{l,i}\| \right] \leq \beta, \quad (7)$$

where β is a known nonnegative constant.

Time-delay systems arise in various engineering practices [1]–[4]. The stability of time-delay systems has attracted considerable research interest [1]–[11]. These studies can be broadly classified into two categories: one based on Lyapunov–Krasovskii functionals [1]–[8]. For example, a necessary and sufficient condition for delay-independent stability of high-order scalar delay systems was given in [1]. The other is based on the distribution of characteristic roots [1]–[4], [9]–[11]. For example, delay-dependent criteria for high-order delay systems were derived in [9] via the argument principle, avoiding computation of characteristic function coefficients.

In practice, parameter uncertainties are unavoidable owing to modeling errors, parameter variations, or unknown environmental conditions [12], [13]. Robust stability explicitly deals with such uncertainties to guarantee properties such as asymptotic stability [14]–[19]. For example, Matveeva [15] studied a class of linear neutral-type systems with periodic coefficients and derived conditions on coefficient perturbations that preserve the exponential stability of the zero solution. In [17], Zhou et al. developed a variable-augmentation approach for robust stability analysis of uncertain time-varying delay systems, resulting in less conservative conditions. In [18], [19], Hu and Hu investigated robust stability of delay systems subject to parametric and delay uncertainties, and sufficient stability conditions were derived via the argument principle in both cases.

Motivated by [18], [19], the present work addresses the robust stability problem for first-order and higher-order time-delay systems subject to unknown parameters and delays. The primary contributions are twofold. First, a sufficient condition is formulated to ensure that asymptotic stability of the nominal system is preserved under perturbations. Second, for the case where the nominal system is unstable, a sufficient condition is developed to guarantee that the perturbed system retains the same count of unstable eigenvalues. Both conditions are derived by means of the argument principle and are straightforward to check.

The remainder of this paper is organized as follows. Section 2 is devoted to the robust stability of first-order time-delay systems, where a review of stability criteria is provided and an upper bound on the unstable characteristic eigenvalues of the perturbed system is established. Section 3 focuses on the robust stability of high-order time-delay systems and establishes the main stability theorems. Section 4 concludes the paper.

Throughout the paper, $\|A\|$ denotes any matrix norm of A .

The notation $\lambda_i(A)$ stands for the i th eigenvalue of A . $\operatorname{Re} z$ denotes the real part of a complex number z .

II. ROBUST STABILITY OF THE FIRST-ORDER TIME-DELAY SYSTEMS

This section first recalls the stability results of [9] for system (2), and then derives an upper bound on the unstable

eigenvalues of the perturbed system (1).

Define the characteristic function of system (2) as

$$P_1(s) = \det \left[sI - A_0 - \sum_{i=1}^m A_i \exp(-\tau_i s) \right]. \quad (8)$$

The roots of $P_1(s)$ are the eigenvalues of system (2). An eigenvalue s is stable if $\operatorname{Re} s < 0$, and unstable otherwise. As shown in [9], an upper bound on all unstable eigenvalues of this system can be established.

Lemma 2.1. Every unstable eigenvalues of (2) satisfies

$$|s| \leq r_1,$$

where

$$r_1 = \|\Delta A_0\| + \sum_{i=1}^m \|\Delta A_i\|.$$

Remark 1. All unstable eigenvalues s of system (2) are bounded and confined to semicircular region

$$\Omega_1 = \{s : |s| \leq r_1, \operatorname{Re} s \geq 0\},$$

with r_1 determined by Lemma 2.1. Let Γ_1 denote the boundary of Ω_1 .

Based on the argument principle, a necessary and sufficient condition for the stability of system (2) is established in the following lemma.

Lemma 2.2. The system (2) is asymptotically stable if and only if

$$P_1(s) \neq 0 \quad (s \in \Gamma_1)$$

and

$$\Delta_{\Gamma_1} \arg P_1(s) = 0,$$

in which $\Delta_{\Gamma_1} \arg P_1(s)$ stands for the change of the argument of $P_1(s)$ along the boundary Γ_1 .

The number of unstable eigenvalues of system (2) is determined by the following lemma.

Lemma 2.3. For system (2), if

$$P_1(s) \neq 0 \quad (s \in \Gamma_1)$$

and

$$\frac{1}{2\pi} \Delta_{\Gamma_1} \arg P_1(s) = z,$$

then system (2) has z unstable eigenvalues.

Remark 2. On the basis of Lemmas 2.2 and 2.3, a numerical algorithm for stability of system (2) was proposed in [9]. We next derive an upper bound on the unstable eigenvalues of the perturbed system (1), whose characteristic function is

$$\tilde{P}_1(s) = \det \left[sI - (A_0 + \Delta A_0) - \sum_{i=1}^m (A_i + \Delta A_i) \exp(-\tilde{\tau}_i s) \right]. \quad (9)$$

Following the proof of Lemma 2.1, a corresponding lemma for the perturbed system can be established.

Lemma 2.4. All unstable eigenvalues of the perturbed system (1) subject to (3) and (4) satisfy

$$|s| \leq r_1 + \alpha,$$

where r_1 given in Lemma 2.1.

Proof. Take s to be an unstable eigenvalue of system (1), that is, $\tilde{P}_1(s) = 0$ with $\operatorname{Re} s \geq 0$. It then follows from Lemma 2.1 that

$$\begin{aligned} |s| &\leq \|A_0 + \Delta A_0\| + \sum_{i=1}^m \|A_i + \Delta A_i\| \\ &\leq \|A_0\| + \|\Delta A_0\| + \sum_{i=1}^m \|A_i\| + \sum_{i=1}^m \|\Delta A_i\| \\ &\leq \left(\|A_0\| + \sum_{i=1}^m \|A_i\| \right) + \left(\|\Delta A_0\| + \sum_{i=1}^m \|\Delta A_i\| \right) \\ &\leq r_1 + \alpha. \end{aligned}$$

Remark 3. Under conditions (3) and (4), all unstable eigenvalues of system (1) are bounded and confined to the semicircular region

$$\Omega_2 = \{s : |s| \leq r_2, \operatorname{Re} s \geq 0\},$$

where $r_2 = r_1 + \alpha$ and r_1 is specified in Lemma 2.1.

Now we denote the boundary of the closed semicircular region Ω_2 as Γ_2 . Next, we will present the main results of this section.

Theorem 2.1. Suppose system (2) is asymptotically stable. If

$$\max_{s \in \Gamma_2} \left[\sum_{i=1}^m \|s R_1^{-1}(s) A_i\| \alpha_i + \|R_1^{-1}(s)\| \alpha \right] < 1 \quad (10)$$

holds, then system (1) under (3) and (4) is asymptotically stable, where $R_1(s)$ is defined by

$$R_1(s) = sI - A_0 - \sum_{i=1}^m A_i \exp(-\tilde{\tau}_i s). \quad (11)$$

Proof. Let

$$E_1(s) = \sum_{i=1}^m A_i (\exp(-\tilde{\tau}_i s) - \exp(-\tau_i s))$$

and

$$E_2(s) = \Delta A_0 + \sum_{i=1}^m \Delta A_i \exp(-\tilde{\tau}_i s).$$

From (8) and (11), $P_1(s) = \det R_1(s)$. The asymptotic stability of system (2) ensures $P_1(s) \neq 0$ and hence the existence of $R_1^{-1}(s)$ for all $s \in \Gamma_2$. Then, by (9), we obtain for $s \in \Gamma_2$,

$$\begin{aligned} \tilde{P}_1(s) &= \det \left[sI - (A_0 + \Delta A_0) \right. \\ &\quad \left. - \sum_{i=1}^m (A_i + \Delta A_i) \exp(-\tilde{\tau}_i s) \right] \end{aligned}$$

$$\begin{aligned} &= \det \left[sI - A_0 - \sum_{i=1}^m A_i \exp(-\tilde{\tau}_i s) \right. \\ &\quad \left. - \Delta A_0 - \sum_{i=1}^m \Delta A_i \exp(-\tilde{\tau}_i s) \right] \\ &= \det \left[sI - A_0 - \sum_{i=1}^m A_i \exp(-\tau_i s) \right. \\ &\quad \left. - \sum_{i=1}^m A_i (\exp(-\tilde{\tau}_i s) - \exp(-\tau_i s)) \right. \\ &\quad \left. - \Delta A_0 - \sum_{i=1}^m \Delta A_i \exp(-\tilde{\tau}_i s) \right] \\ &= \det[R_1(s) - E_1(s) - E_2(s)] \\ &= \det R_1(s) \det [I - R_1^{-1}(s) E_1(s) \\ &\quad - R_1^{-1}(s) E_2(s)] \\ &= P_1(s) \det[I - Q_1(s)]. \end{aligned}$$

Thus, we deduce

$$\begin{aligned} \tilde{P}_1(s) &= P_1(s) \det[I - Q_1(s)] \\ &= P_1(s) \prod_{j=1}^n [1 - \lambda_j(Q_1(s))]. \end{aligned} \quad (12)$$

where $Q_1(s) = R_1^{-1}(s)(E_1(s) + E_2(s))$.

It follows from Lemma 2.4 and Remark 3 that all unstable eigenvalues s of system (1) subject to (3) and (4) are contained in Ω_2 . Consequently, the stability of system (1) reduces to checking whether it has any eigenvalue in Ω_2 . For $s \in \Gamma_2$, condition (10) together with $\operatorname{Re} s \geq 0$ yields

$$\begin{aligned} |\lambda_j(Q_1(s))| &\leq \|Q_1(s)\| \\ &= \|R_1^{-1}(s)(E_1(s) + E_2(s))\| \\ &\leq \|R_1^{-1}(s)E_1(s)\| + \|R_1^{-1}(s)E_2(s)\| \\ &= \left\| R_1^{-1}(s) \left[\sum_{i=1}^m A_i (\exp(-\tilde{\tau}_i s) - \exp(-\tau_i s)) \right] \right\| \\ &\quad + \left\| R_1^{-1}(s) \left[\Delta A_0 + \sum_{i=1}^m \Delta A_i \exp(-\tilde{\tau}_i s) \right] \right\| \\ &\leq \left\| R_1^{-1}(s) \left[\sum_{i=1}^m A_i s \int_{-\tau_i}^{-\tilde{\tau}_i} \exp(st) dt \right] \right\| \\ &\quad + \left\| R_1^{-1}(s) \left[\Delta A_0 + \sum_{i=1}^m \Delta A_i \exp(-\tilde{\tau}_i s) \right] \right\| \\ &\leq \sum_{i=1}^m \left\| R_1^{-1}(s) \left[A_i s \int_{-\tau_i}^{-\tilde{\tau}_i} \exp(st) dt \right] \right\| \\ &\quad + \|R_1^{-1}(s)\| \left[\|\Delta A_0\| + \sum_{i=1}^m \|\Delta A_i\| \exp(-\tilde{\tau}_i s) \right] \\ &\leq \sum_{i=1}^m \|s R_1^{-1}(s) A_i\| \left| \int_{-\tau_i}^{-\tilde{\tau}_i} \exp(st) dt \right| \\ &\quad + \|R_1^{-1}(s)\| \left[\|\Delta A_0\| + \sum_{i=1}^m \|\Delta A_i\| \exp(-\tilde{\tau}_i s) \right] \\ &\leq \sum_{i=1}^m \|s R_1^{-1}(s) A_i\| |\tau_i - \tilde{\tau}_i| \\ &\quad + \|R_1^{-1}(s)\| \left[\|\Delta A_0\| + \sum_{i=1}^m \|\Delta A_i\| \right] \end{aligned}$$

$$\begin{aligned} &\leq \sum_{i=1}^m \|sR_1^{-1}(s)A_i\|\alpha_i \\ &\quad + \|R_1^{-1}(s)\|\alpha \\ &\leq \max_{s \in \Gamma_2} \left[\sum_{i=1}^m \|sR_1^{-1}(s)A_i\|\alpha_i \right. \\ &\quad \left. + \|R_1^{-1}(s)\|\alpha \right] \\ &\leq 1. \end{aligned}$$

i.e.,

$$|\lambda_j(Q_1(s))| < 1. \quad (13)$$

Combining this with (12), we obtain for all $s \in \Gamma_2$,

$$\tilde{P}_1(s) \neq 0. \quad (14)$$

Furthermore, for $s \in \Gamma_2$, we obtain

$$\begin{aligned} \Delta_{\Gamma_2} \arg \tilde{P}_1(s) &= \Delta_{\Gamma_2} \arg P_1(s) \\ &\quad + \sum_{j=1}^n \Delta_{\Gamma_2} \arg[1 - \lambda_j(Q_1(s))] \\ &= 0 + \sum_{j=1}^n 0 = 0. \end{aligned} \quad (15)$$

Therefore, by (14), (15), and Lemma 2, the perturbed system (1) under (3) and (4) is asymptotically stable. This completes the proof.

Theorem 2.2. Suppose that the conditions in Lemma 2.3 are satisfied for system (2). Then, provided that (10) holds, the perturbed system (1) subject to (3) and (4) possesses z unstable eigenvalues.

Proof. The proof is analogous to that of Theorem 1. Under the assumptions of Lemma 2.3, all unstable eigenvalues of system (2) are contained in Ω_1 . Consequently, we have $P_1(s) \neq 0$ and $R_1^{-1}(s)$ exists for $s \in \Gamma_2$. For these $s \in \Gamma_2$, we have $\operatorname{Re} s \geq 0$ and $|\exp(-\tau_j s)| \leq 1$ for $j = 1, 2, \dots, m$. It then follows from (10), (12), and (13) that

$$\tilde{P}_1(s) \neq 0 \quad \text{for } s \in \Gamma_2. \quad (16)$$

Furthermore, we compute

$$\begin{aligned} \Delta_{\Gamma_2} \arg \tilde{P}_1(s) &= \Delta_{\Gamma_2} \arg P_1(s) \\ &\quad + \sum_{j=1}^n \Delta_{\Gamma_2} \arg[1 - \lambda_j(Q_1(s))] \\ &= 2\pi z + \sum_{j=1}^n 0 = 2\pi z. \end{aligned} \quad (17)$$

Thus, by (16), (17), and Lemma 2.3, the perturbed system (1) under (3) and (4) has z unstable eigenvalues. This completes the proof.

Remark 4. The theorems above demonstrate that (10) suffices to preserve the stability of system (1) under (3) and (4). Thus, if the nominal system (2) is stable, so is the perturbed system; if unstable, the number of unstable eigenvalues remains the same.

Remark 5. It is worth noting that Theorem 2.1 and Theorem 2.2 encompass two existing results as special cases.

Specifically, when no delay perturbations are present (i.e., $\alpha_i = 0$, $i = 1, 2, \dots, m$), the theorems degenerate into the condition given in [18]; when no matrix perturbations are present (i.e., $\alpha = 0$), they degenerate into the stability conditions reported in [19].

IV. ROBUST STABILITY OF THE HIGH-ORDER TIME-DELAY SYSTEMS

This section recalls the stability results of [10] for the high-order time-delay systems (6), whose characteristic function is

$$P_2(s) = \det \left\{ Is^n + \sum_{l=1}^n \left[A_l s^{(n-l)} + \sum_{i=1}^m B_{l,i} s^{(n-l)} \exp(-\tau_i s) \right] \right\}. \quad (18)$$

System (6) is asymptotically stable if and only if all the characteristic roots of $P_2(s)$ lie in the open left half-plane [10]. An upper bound on the unstable eigenvalues of system (6) was given in [10].

Lemma 3.1. Every unstable eigenvalues s of (6) satisfies

$$|s| \leq r_3,$$

where

$$r_3 = \max\{1, b\}, \quad b = \sum_{l=1}^n \left[\|A_l\| + \sum_{i=1}^m \|B_{l,i}\| \right].$$

Remark 6. All unstable eigenvalues s of system (6) are bounded and contained in a semicircular region

$$\Omega_3 = \{s : |s| \leq r_3, \operatorname{Re} s \geq 0\},$$

with r_3 determined by Lemma 3.1. Let Γ_3 denote the boundary of Ω_3 .

Using the argument principle, the following lemma offers a necessary and sufficient condition for the stability of system (6).

Lemma 3.2. The high-order time-delay system (6) is asymptotically stable if and only if

$$P_2(s) \neq 0 \quad (s \in \Gamma_3)$$

and

$$\Delta_{\Gamma_3} \arg P_2(s) = 0,$$

where $\Delta_{\Gamma_3} \arg P_2(s)$ stands for the change of the argument of $P_2(s)$ along the boundary Γ_3 .

The number of unstable eigenvalues of system (6) is determined by the following lemma.

Lemma 3.3. For the high-order time-delay system (6), if

$$P_2(s) \neq 0 \quad (s \in \Gamma_3)$$

and

$$\frac{1}{2\pi} \Delta_{\Gamma_3} \arg P_2(s) = z,$$

then system (6) possesses z unstable eigenvalues.

Remark 7. A numerical algorithm for verifying the stability of system (6) was developed in [10] based on Lemma 3.2. An upper bound on all unstable eigenvalues of the perturbed system (5) is now derived. For this system subject to (4) and (7), the characteristic function is given by

$$\tilde{P}_2(s) = \det \left\{ I s^n + \sum_{l=1}^n \left[(A_l + \Delta A_l) s^{(n-l)} + \sum_{i=1}^m (B_{l,i} + \Delta B_{l,i}) s^{(n-l)} \exp(-\tilde{\tau}_i s) \right] \right\}. \quad (19)$$

By the same argument as in Lemma 3.1, we derive a counterpart lemma for the perturbed system (5).

Lemma 3.4. For the perturbed system (5) with (4) and (7), the following bound holds for all unstable eigenvalues s

$$|s| \leq r_4,$$

where

$$r_4 = \max\{1, \tilde{b}\}, \quad \tilde{b} = r_3 + \beta.$$

Proof. Let s to be an unstable eigenvalue of system (5), i.e., $\tilde{P}_2(s) = 0$ with $\operatorname{Re} s \geq 0$. For the case $|s| \geq 1$, it follows from (19) that

$$\begin{aligned} \tilde{P}_2(s) &= \det \left\{ I s^n + \sum_{l=1}^n \left[(A_l + \Delta A_l) s^{(n-l)} + \sum_{i=1}^m (B_{l,i} + \Delta B_{l,i}) s^{(n-l)} \exp(-\tilde{\tau}_i s) \right] \right\} \\ &= \det s^{n-1} \left\{ I s + \sum_{l=1}^n \left[(A_l + \Delta A_l) s^{(1-l)} + \sum_{i=1}^m (B_{l,i} + \Delta B_{l,i}) s^{(1-l)} \exp(-\tilde{\tau}_i s) \right] \right\}. \end{aligned}$$

We define the matrix

$$W(s) = - \sum_{l=1}^n \left[(A_l + \Delta A_l) s^{(1-l)} + \sum_{i=1}^m (B_{l,i} + \Delta B_{l,i}) s^{(1-l)} \exp(-\tilde{\tau}_i s) \right].$$

Equation (19) can be rewritten as

$$\tilde{P}_2(s) = \det s^{n-1} [I s - W(s)],$$

which implies

$$\det[I s - W(s)] = 0.$$

Hence s is an eigenvalue of $W(s)$, so there exists some j ($1 \leq j \leq n$) such that

$$s = \lambda_j(W(s)). \quad (20)$$

Since $|s| \geq 1$ and $\operatorname{Re} s \geq 0$, we obtain

$$|s^{1-l}| \leq 1, \quad l = 1, 2, \dots, n$$

and

$$|\exp(-\tilde{\tau}_j s)| \leq 1, \quad j = 1, 2, \dots, m.$$

From (20), it follows that

$$|s| = |\lambda_j(W(s))| \leq \|W(s)\|.$$

By the definition of $W(s)$, we derive

$$\begin{aligned} \|W(s)\| &= \left\| \sum_{l=1}^n (A_l + \Delta A_l) s^{(1-l)} + \sum_{i=1}^m (B_{l,i} + \Delta B_{l,i}) s^{(1-l)} \exp(-\tilde{\tau}_i s) \right\| \\ &\leq \sum_{l=1}^n \left[\|A_l\| + \sum_{i=1}^m \|B_{l,i}\| \right] + \sum_{l=1}^n \left[\|\Delta A_l\| + \sum_{i=1}^m \|\Delta B_{l,i}\| \right] \\ &\leq r_3 + \beta. \end{aligned}$$

Thus, we have $|s| \leq r_3 + \beta$. Therefore, for any unstable eigenvalue s , we conclude

$$|s| \leq \max\{1, \tilde{b}\},$$

holds, where $\tilde{b} \leq r_3 + \beta$.

Remark 8. For system (5) under (4) and (7), all unstable eigenvalues s are bounded and confined to the semicircular region

$$\Omega_4 = \{s : |s| \leq r_4, \operatorname{Re} s \geq 0\},$$

where $r_4 = \max\{1, r_3 + \beta\}$ with r_3 is from Lemma 3.1.

Now we denote the boundary of the closed semicircular region Ω_4 as Γ_4 . We next give the main stability theorems for system (5) under (4) and (7).

Theorem 2.1. Suppose system (6) is asymptotically stable. If

$$\max_{s \in \Gamma_4} \left[\sum_{l=1}^n \sum_{i=1}^m \|s^{n-l+1} R_2^{-1}(s) B_{l,i}\| \alpha_i + \|R_2^{-1}(s)\| \beta r_4^{n-1} \right] < 1 \quad (21)$$

holds, then system (5) satisfying (4) and (7) is asymptotically stable, where the matrix $R_2(s)$ is defined by

$$R_2(s) = I s^n + \sum_{l=1}^n \left[A_l s^{(n-l)} + \sum_{i=1}^m B_{l,i} s^{(n-l)} \exp(-\tau_i s) \right]. \quad (22)$$

Proof. Let

$$E_3(s) = \sum_{l=1}^n \sum_{i=1}^m B_{l,i} s^{n-l} (\exp(-\tilde{\tau}_i s) - \exp(-\tau_i s))$$

and

$$E_4(s) = \sum_{l=1}^n \left[\Delta A_l s^{(n-l)} + \sum_{i=1}^m \Delta B_{l,i} s^{(n-l)} \exp(-\tilde{\tau}_i s) \right].$$

It follows from (18) and (22) that $P_2(s) = \det R_2(s)$. Since system (6) is asymptotically stable, $P_2(s) \neq 0$ and consequently $R_2^{-1}(s)$ exists for all $s \in \Gamma_2$. Then, from (19), we have for $s \in \Gamma_4$,

$$\begin{aligned} \tilde{P}_2(s) &= \det \left\{ Is^n + \sum_{l=1}^n \left[(A_l + \Delta A_l) s^{(n-l)} \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^m (B_{l,i} + \Delta B_{l,i}) s^{(n-l)} \exp(-\tilde{\tau}_i s) \right] \right\} \\ &= \det \left\{ Is^n + \sum_{l=1}^n \left[A_l s^{(n-l)} \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^m B_{l,i} s^{(n-l)} \exp(-\tau_i s) \right] \right. \\ &\quad \left. + \sum_{l=1}^n \sum_{i=1}^m B_{l,i} s^{n-l} (\exp(-\tilde{\tau}_i s) - \exp(-\tau_i s)) \right. \\ &\quad \left. + \sum_{l=1}^n \left[\Delta A_l s^{(n-l)} \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^m \Delta B_{l,i} s^{(n-l)} \exp(-\tilde{\tau}_i s) \right] \right\} \\ &= \det [R_2(s) - E_3(s) - E_4(s)] \\ &= \det R_2(s) \det \left[I - R_2^{-1}(s) E_3(s) \right. \\ &\quad \left. - R_2^{-1}(s) E_4(s) \right] \\ &= P_2(s) \det [I - Q_2(s)]. \end{aligned}$$

We thus obtain

$$\begin{aligned} \tilde{P}_2(s) &= P_2(s) \det [I - Q_2(s)] \\ &= P_2(s) \prod_{j=1}^n [1 - \lambda_j(Q_2(s))]. \end{aligned} \quad (23)$$

where $Q_2(s) = R_2^{-1}(s)(E_3(s) + E_4(s))$.

Lemma 3.4 and Remark 8 imply that all unstable eigenvalues of system (5) under (4) and (7) lie in Ω_4 . Therefore, the stability of system (5) reduces to checking for eigenvalues in Ω_4 . For $s \in \Gamma_4$, from condition (21) and $\operatorname{Re} s \geq 0$, we have

$$\begin{aligned} |\lambda_j(Q_2(s))| &\leq \|Q_2(s)\| \\ &= \|R_2^{-1}(s)(E_3(s) + E_4(s))\| \\ &\leq \|R_2^{-1}(s)E_3(s)\| + \|R_2^{-1}(s)E_4(s)\| \\ &= \left\| R_2^{-1}(s) \sum_{l=1}^n \sum_{i=1}^m B_{l,i} s^{n-l} \right\| \\ &\quad \left\| (\exp(-\tilde{\tau}_i s) - \exp(-\tau_i s)) \right\| \\ &\quad + \left\| R_2^{-1}(s) \sum_{l=1}^n \left[\Delta A_l s^{(n-l)} \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^m \Delta B_{l,i} s^{(n-l)} \exp(-\tilde{\tau}_i s) \right] \right\| \\ &\leq \left\| \sum_{l=1}^n \sum_{i=1}^m R_2^{-1}(s) B_{l,i} s^{n-l} \right\| \\ &\quad \left\| (\exp(-\tilde{\tau}_i s) - \exp(-\tau_i s)) \right\| \end{aligned}$$

$$\begin{aligned} &+ \|R_2^{-1}(s)\| \sum_{l=1}^n \left[\|\Delta A_l\| s^{n-l} \right. \\ &\quad \left. + \sum_{i=1}^m \|\Delta B_{l,i}\| s^{n-l} \|\exp(-\tilde{\tau}_i s)\| \right] \\ &\leq \left\| \sum_{l=1}^n \sum_{i=1}^m R_2^{-1}(s) B_{l,i} s^{n-l} s \right\| \\ &\quad \left\| \int_{-\tau_i}^{-\tilde{\tau}_i} \exp(st) dt \right\| \\ &\quad + \|R_2^{-1}(s)\| \sum_{l=1}^n \left[\|\Delta A_l\| s^{n-l} \right. \\ &\quad \left. + \sum_{i=1}^m \|\Delta B_{l,i}\| s^{n-l} \|\exp(-\tilde{\tau}_i s)\| \right] \\ &\leq \sum_{l=1}^n \sum_{i=1}^m \|R_2^{-1}(s) B_{l,i} s^{n-l} s\| |\tilde{\tau}_i - \tau_i| \\ &\quad + \|R_2^{-1}(s)\| \left[\sum_{l=1}^n \|\Delta A_l\| \right. \\ &\quad \left. + \sum_{l=1}^n \sum_{i=1}^m \|\Delta B_{l,i}\| \right] r_4^{n-1} \\ &\leq \sum_{l=1}^n \sum_{i=1}^m \|s^{n-l+1} R_2^{-1}(s) B_{l,i}\| \alpha_i + \|R_2^{-1}(s)\| \beta r_4^{n-1} \\ &\leq \max_{s \in \Gamma_4} \left[\sum_{l=1}^n \sum_{i=1}^m \|s^{n-l+1} R_2^{-1}(s) B_{l,i}\| \alpha_i \right. \\ &\quad \left. + \|R_2^{-1}(s)\| \beta r_4^{n-1} \right] \\ &< 1. \end{aligned}$$

i.e.,

$$|\lambda_j(Q_2(s))| < 1. \quad (24)$$

This and (23) imply that for all $s \in \Gamma_4$,

$$\tilde{P}_2(s) \neq 0. \quad (25)$$

In addition, for $s \in \Gamma_2$, we obtain

$$\begin{aligned} \Delta_{\Gamma_4} \arg \tilde{P}_2(s) &= \Delta_{\Gamma_4} \arg P_2(s) \\ &\quad + \sum_{j=1}^n \Delta_{\Gamma_4} \arg [1 - \lambda_j(Q_2(s))] \\ &= 0 + \sum_{j=1}^n 0 = 0. \end{aligned} \quad (26)$$

Combining (25), (26), and Lemma 6, we obtain that the perturbed system (5) under (4) and (7) is asymptotically stable. This completes the proof.

Theorem 3.2. Assume that system (6) satisfies the conditions of Lemma 3.3. Then, if condition (21) is also satisfied, the perturbed system (5) under (4) and (7) has exactly z unstable eigenvalues.

Proof. The proof is based on a technique similar to that used in Theorem 3.1. Under the conditions of Lemma 3.3, all unstable eigenvalues of system (6) lie in Ω_4 . Hence, $P_2(s) \neq 0$ and $R_2^{-1}(s)$ exists for $s \in \Gamma_4$. For $s \in \Gamma_4$, we note that $\operatorname{Re} s \geq 0$ and $|\exp(-\tau_j s)| \leq 1$ for $j = 1, 2, \dots, m$. It then follows from (21), (23), and (24) that

$$\tilde{P}_2(s) \neq 0 \quad \text{for } s \in \Gamma_4. \quad (27)$$

Moreover, for $s \in \Gamma_4$, we obtain

$$\begin{aligned}\Delta_{\Gamma_4} \arg \tilde{P}_2(s) &= \Delta_{\Gamma_4} \arg P_2(s) \\ &\quad + \sum_{j=1}^n \Delta_{\Gamma_4} \arg[1 - \lambda_j(Q_1(s))] \\ &= 2\pi z + \sum_{j=1}^n 0 = 2\pi z.\end{aligned}\quad (28)$$

Combining (27), (28), and Lemma 7, we obtain that the perturbed system (5) under (4) and (7) has z unstable eigenvalues. The proof is thus complete.

Remark 9. The preceding theorems demonstrate that (21) suffices to preserve the stability of system (5) under (4) and (7). Thus, if the nominal system (6) is stable, so is the perturbed system; if unstable, the number of unstable eigenvalues remains unchanged.

Remark 10. The Theorems 3.1 and 3.2 generalize several existing stability criteria. In the absence of delay uncertainties (i.e., $\alpha_i = 0$, $i = 1, 2, \dots, m$), they reduce to the matrix-perturbation results of [18]; in the absence of matrix uncertainties (i.e., $\beta = 0$), they reduce to the delay-perturbation results of [19]. More importantly, unlike [18], which is confined to first-order systems, our theorems further extend these results to high-order delay systems.

V. CONCLUSION

This paper addresses the robust stability of linear time-delay systems with unknown parameters and delays. By exploiting the distribution of characteristic roots and the argument principle, an upper bound on the unstable eigenvalues of the perturbed system is established, and sufficient conditions are derived for both first-order and high-order systems. The proposed method avoids computing characteristic function coefficients, resulting in low computational cost and easy verification. Furthermore, the derived conditions account for parameter and delay uncertainties simultaneously, guaranteeing that the perturbed system either preserves asymptotic stability or retains the same number of unstable eigenvalues. The results extend several existing criteria to high-order systems within a unified framework.

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