Recursive Consensus Estimation for Networked Systems With Random Delays: An Edge-Based Event-Triggered Scheme

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ABSTRACT This paper researches recursive consensus filtering in networked systems affected by random transmission delays under edge-based event-triggered schemes. The study introduces a dynamic edge-level triggering rule to streamline data exchange and alleviate the network's communication burden. The information incompleteness due to delays is address by transforming the time-delayed observation system into an observation system with zero-time delay through virtualization and rearrangement of the observation sequence. Based on this, a novel consensus filter is proposed, with consensus gain parameters optimized through minimizing the upper bound of the estimation error covariance. Ultimately, this research sheds light on innovative strategies for enhancing consensus filtering in networked systems with random transmission delays.

INDEX TERMS Consensus estimation, edge-based event-triggered schemes, random delays, virtualization and rearrangement.

I. INTRODUCTION

In recent years, there has been a surge of interest in networked systems due to their diverse applications across various sectors such as environmental monitoring, target tracking, smart grids, navigation, and industrial automation [1][2][3]. Compared with traditional point-to-point estimation architectures, networked sensor systems possess advantages, for instance, they are flexible to deploy, easy to be installed, have low maintenance cost, and boast strong scalability [4]. In such systems, multiple sensor nodes collect local measurements and exchange information through communication networks so as to estimate the state of a dynamic plant. Consequently, reliable state reconstruction is a fundamental task in networked-system analysis and design [5][6]. According to the way information is processed, estimation strategies over sensor networks can generally be divided into centralized and distributed approaches. A centralized architecture forwards the measurements collected by every sensor to one fusion center for global processing. Although this architecture can make full use of the available measurement information, it usually requires high communication bandwidth and is sensitive to the failure of the fusion center [7]. By contrast, distributed estimation allows each node to generate its own state estimate by using local measurements and information received from neighboring nodes. Distributed estimation is therefore better suited to many wireless sensor networks because it improves communication efficiency and energy saving while robustness and scalability [8]. However, the introduction of communication networks also brings various network-induced phenomena, including packet dropouts, random delays, limited bandwidth, fading channels, and data congestion [9][10][11][12]. Among these phenomena, random transmission delays are especially important in networked estimation systems [13][14]. In practical wireless networks, when the amount of transmitted data exceeds the communication capacity, data packets may wait in buffers before being delivered to the estimator. As a result, the received measurement may correspond to a previous system state rather than the current one. Such delayed information may lead to incomplete real-time measurement information and degrade the estimation accuracy. In addition, packet dropouts may occur when the delayed data cannot be successfully received. Therefore, it is meaningful to establish a unified model that can simultaneously describe delay-free transmission, delayed transmission, and packet dropout. To deal with delayed measurements, literature reported several approaches, including the state augmentation method, the linear matrix inequality (LMI) method, and the measurement reorganization method [15][16][4][17]. Among them, the state augmentation method is straightforward, but it usually increases the system dimension and leads to additional computational burden.

The LMI-based method can provide sufficient conditions for stability or performance analysis, but the obtained results may be conservative and are often not convenient for recursive implementation. The measurement reorganization method provides another promising way to handle transmission delays by rearranging the delayed observations into an equivalent delay-free observation sequence. Nevertheless, how to fully utilize incomplete measurement information caused by random delays while maintaining a recursive distributed estimation structure is still a challenging issue. On the other hand, communication resources in wireless sensor networks are usually limited. If each node transmits information to all its neighbors at every time instant, a large amount of redundant information will be generated, which may increase communication cost and energy consumption. Event-triggered schemes have been widely studied as an effective way to reduce unnecessary data transmissions [18][19][20][21]. In traditional node-based event-triggered schemes [22], [23], nodes transmit information to all neighboring nodes simultaneously once a triggering condition is met [24]. Although this strategy can reduce the communication frequency, it may still transmit redundant information to some neighbors. To overcome this limitation, edge-based event-triggered schemes have been proposed [5], [25], [26], where the triggering decision is made for each communication edge. Compared with node-based schemes, edge-based schemes enable each node to communicate with different neighbors at varying triggering instants, providing a more flexible asynchronous communication structure. Motivated by the above observations, this paper studies recursive consensus estimation for a class of linear networked systems and take involves implementing an edge-based event-triggered scheme to solve random transmission delays issue. The random transmission status which is modeled by Bernoulli variables enables the received measurement model to describe delay-free transmission, delayed transmission, and packet dropout in a unified probabilistic framework. To overcome the incomplete information problem caused by random delays, the delayed measurement sequence is decomposed into virtual measurement sequences and then rearranged into an equivalent measurement sequence without delay. According to the rearranged observations, a local minimum variance consensus filter is developed. Furthermore, a dynamic sender–receiver event-triggered mechanism is developed for individual communication edges, which effectively controls redundant information exchange between neighboring nodes while maintaining the estimation performance at a high standard. The main contributions are as follows:

1. A unified probabilistic measurement technique is constructed for networked systems which has random transmission delays. By using Bernoulli random variables, the proposed model can simultaneously characterize delay-free transmission, delayed transmission, and packet dropout, which makes the considered system model more suitable for practical networked estimation scenarios.
2. A virtual measurement rearrangement method is devised to tackle the issue of incomplete data due to random delays. The delayed measurements are first decomposed into virtual measurement sequences and then reorganized into an equivalent delay-free measurement sequence. This avoids the high-dimensional state augmentation procedure and provides a convenient basis for recursive filter design.
3. A novel algorithm is developed for recursive consensus estimation in a dynamic edge-based sender-receiver event-triggered system. By utilizing relative state information and a dynamic threshold variable, redundant transmissions are minimized effectively. Through rearranged measurements, the algorithm calculates optimal local minimum-variance filter gain and consensus gain to minimize estimation errors. This approach improves efficiency and accuracy in consensus estimation. The paper is structured as follows, section II presents the randomly delayed networked system model and describes the edge-based event-triggering mechanism. Section III presents the virtual measurement rearrangement method and the preliminary results. Section IV develops the recursive consensus filter and derives the corresponding gain matrices. Section V numerical results will be presented to showcase the effectiveness of the proposed approach. Finally, Section VI provides a conclusion for this paper.

II. PROBLEM FORMULATION

The wireless network's structure is defined by the graph $G = (V, \mathcal{E})$, with V representing the set of nodes and $\mathcal{E}$ representing the set of edges (communication links). In a directed graph, an edge (i, j) (pointing from i to j) indicates the flow of information from node i to node j, with node i considered as an in-neighbor node of node j. Besides, N_j represents the set of all in-neighbors of node j. In an undirected graph, if node i and node j are both in-neighbors, which means that there is an edge $(i, j) \in \mathcal{E}$, a weight a_{ij} is assigned to each edge $(i, j) \in \mathcal{E}$, with n_i representing the number of in-neighbors of node i. This study considers both directed and undirected network topologies within a general graph framework. Consider the following networked system model:

$$\begin{cases} x(z+1) = A(z)x(z) + B(z)w(z), \\ y_i(z) = C_i(z)x(z) + D_i(z)v_i(z) \end{cases} \tag{1}$$

where $x(z)$ denotes the state vector and $y_i(z)$ is the observation output vector. $w(z)$ and $v_i(z)$ are random disturbances. $A(z)$, $B(z)$, $C_i(z)$, and $D_i(z)$ are known matrices with compatible dimensions. The random variables $w(z)$ and $v_i(z)$ are mutually independent, i.e.

$$\hat{\mathbb{E}}\left\{\begin{bmatrix} w(z) \\ v_i(z) \end{bmatrix} \begin{bmatrix} w^T(z) & v_i^T(z) \end{bmatrix}\right\} = \begin{bmatrix} Q(z) & 0 \\ 0 & R_i(z) \end{bmatrix} \tag{2}$$

where $Q(z)$ and $R_i(z)$ are known parameters and have appropriate dimensions. Assumption 1: The primary conditions of system (1) satisfy

$$\hat{\mathbb{E}}\{x(0)\} = x_0, \quad \hat{\mathbb{E}}\{(x(0) - x_0)(x(0) - x_0)^T\} = P_0 \tag{3}$$

where x_0 is a known vector, and P_0 is a known shift-varying matrix of appropriate dimension. Moreover, $x(0)$ is uncorrelated with $w(z)$ and $v_i(z)$. Due to network bandwidth limitations, when the data injection rate exceeds the communication link capacity, network congestion often leads to random transmission delays. The estimator receives measurements described by the following model:

$$\bar{y}_i(z) = \bar{\zeta}_1(z)y_i(z) + \bar{\zeta}_2(z)\big(1 - \bar{\zeta}_1(z)\big)y_i(z - \varsigma) \tag{4}$$

where $\bar{y}_i(z)$ is the observed output at the estimator side and ς is the random delay. The random variables $\bar{\zeta}_c(z)$ are mutually uncorrelated and follow a Bernoulli distribution with

$$\begin{cases} \text{Prob}\{\bar{\zeta}_c(z) = 1\} = \varpi_c(z), \\ \text{Prob}\{\bar{\zeta}_c(z) = 0\} = 1 - \varpi_c(z) \end{cases} \tag{5}$$

Remark 1: The random variables $\bar{\zeta}_1(z)$ and $\bar{\zeta}_2(z)$ are used to characterize whether delay or packet loss occurs, specifically, as stated in Appendix A, Eq. (A1). Therefore, the proposed description can simultaneously characterize delay-free transmission, delayed transmission, and packet loss within a unified probabilistic model, making it more general.
Remark 2: Random transmission delays are common network-induced phenomena in networked estimation systems. In practical wireless sensor networks, measurement data are transmitted through communication channels with limited bandwidth and unstable communication quality. When the data traffic exceeds the available channel capacity, data packets may have to wait in buffers before being delivered to the estimator, which gives rise to random transmission delays. Furthermore, transmission distance, channel congestion, and unreliable wireless environments further exacerbate the random nature of the delay. Random delays have a direct influence on the estimation problem. The estimator cannot always obtain real-time state information. This leads to incomplete measurement information, which may degrade the estimation accuracy. Under the sender–receiver ETS, each communication edge is equipped with a trigger that decides whether the sender's state information is delivered to the receiver. Both sending and receiving nodes jointly evaluate the triggering condition for each edge. Let $\hat{x}^j_{i,\text{pri}}(z)$ denote node j's real-time estimation of its neighboring node i's prior state estimate. Initially, the storer in the data packet scheduler calculates $\hat{x}^j_{i,\text{pri}}(z)$ and $\hat{x}^i_{j,\text{pri}}(z)$ using stored information. The event generator then checks whether the relative state mismatch satisfies the triggering rule assigned to edge (i, j). Accordingly, $\tilde{x}^j_{i,\text{pri}}(z)$ is updated according to one of the following two cases, as stated in Appendix A, Eq. (A2). At last, $\tilde{x}^j_{i,\text{pri}}(z)$ is transmitted to receiver node j, updated in the storage unit, and then returned to sender node i. To be more specific, define the triggering-time sequence associated with edge (i, j) as follows:

$$0 = z_0^{i\to j} \le z_1^{i\to j} \le \cdots \le z_s^{i\to j} \le \cdots \tag{6}$$

The real-time estimation satisfies the following recursive calculation

$$\hat{x}^j_{i,\text{pri}}(z) = A^{\,z - z_s^{i\to j}}\left(z_s^{i\to j}\right)\hat{x}_{i,\text{pri}}\left(z_s^{i\to j}\right) \tag{7}$$

within the storage unit shared by nodes i and j. The communication error from node i to node j is defined as

$$e^j_{i,\text{pri}}(z) \triangleq \hat{x}^j_{i,\text{pri}}(z) - \hat{x}_{i,\text{pri}}(z) \tag{8}$$

The relative state information associated with edge (i, j), linking sender node i and receiving node j, is defined in Appendix A, Eq. (A3). This quantity characterizes the bidirectional information mismatch between sender node i and receiver node j. Based on the relative state information $\xi_{ij,\text{pri}}(z)$, the event generator associated with edge (i, j) is defined as follows:

$$\mathcal{F}\big(\xi_{ij,\text{pri}}(z)\big) = \gamma_{ij}\big(a_{ij}\|\xi_{ij,\text{pri}}(z)\| - \iota_{ij}\big) \tag{9}$$

where γ_{ij} and ι_{ij} are given parameters. The forthcoming triggering instant for edge (i, j) is specified by the event generator according to the rule in Appendix A, Eq. (A4), where the dynamic threshold variable $\delta_{ij}(z)$ is governed by the following dynamics:

$$\delta_{ij}(z + 1) = \rho_{ij}\delta_{ij}(z) - \frac{1}{\gamma_{ij}}\mathcal{F}\big(\xi_{ij,\text{pri}}(z)\big) \tag{10}$$

where ρ_{ij} and γ_{ij} are the pre-assigned scalars, and $\delta_{ij}(0)$ denotes the prescribed initial value of the positive dynamic variable $\delta_{ij}(z) > 0$.
Remark 3: The dynamic event-triggered mechanism is employed to suppress redundant transmissions while improving estimation accuracy in the networked estimation process. Unlike a static event-triggering rule with a constant threshold determined solely by preset parameters, the dynamic mechanism uses an auxiliary variable to adjust the triggering condition according to the evolution of the system information. Therefore, when the estimation disagreement is small, the triggering condition becomes harder to satisfy, and redundant transmissions can be avoided. When the disagreement becomes large, the communication can still be activated to preserve the estimation performance. In this sense, the dynamic event-triggered mechanism provides a more flexible tradeoff between communication cost and estimation accuracy.
Remark 4: As indicated by (9), the event generator function is primarily governed by the system parameter $\|\xi_{ij,\text{pri}}(z)\|$, with parameter a_{ij} assigning weight to $\|\xi_{ij,\text{pri}}(z)\|$. In (9), the parameter ι_{ij} serves as a balancing factor to regulate the triggering frequency. When the number of triggers is excessively high, increasing ι_{ij} can significantly reduce the triggering rate, albeit at the cost of a slight increase in estimation error. The parameter γ_{ij} acts as a scaling factor; a larger γ_{ij} results in a higher triggering frequency. Furthermore, (A8) shows that ρ_{ij} is bounded within $(0, 1)$. According to (10), as ρ_{ij} approaches the upper limit of 1, the number of triggers tends to decrease.

III. PRELIMINARIES

A. VIRTUALIZATION AND REARRANGEMENT OF DELAYED OBSERVATION SEQUENCE

As indicated by (4), random delays cause measurements to fail to capture the current system state, leading to incomplete real-time information for state estimation systems. To mitigate this issue, we propose decomposing delayed measurements into multiple virtual measurement sequences, each governed by an independent random delay process. The corresponding equation is provided in Appendix A as Eq. (A5).

Remark 5: An inspection of the fictitious sequence (A5) reveals that each horizon contains a single component, $y_i^c(z)$ $(c = 1, 2)$, which is regulated by $\zeta_c(z)$. This decomposition of delayed measurements into distinct fictitious streams serves to circumvent the prohibitive complexity typically associated with state augmentation. Considering the existence of random delays, the measurement received in each horizon of (4) remains indeterminate. While (A5) introduces a fictitious observation sequence model, the composition of these sequences differs from one horizon to another. Based on this, the original delayed measurement sequence can be formulated as:

1) for $0 \le z < \varsigma$,

$$\tilde{y}_i^1(z) = y_i^1(z) \tag{11}$$

2) for $z \ge \varsigma$,

$$\tilde{y}_i^2(z) = \begin{bmatrix} y_i^1(z) \\ y_i^2(z) \end{bmatrix} \tag{12}$$

In this context, all possible delayed observations within each time horizon are organized into a sequence, referred to as the original delayed fictitious measurement sequence. However, since $\tilde{y}_i^2(z)$ itself suffers from delays, the state estimation information becomes incomplete. This issue renders invalid both the filter structure derived from projection algorithms and any pre-defined filter architectures. Therefore, this paper aims to fully utilize this incomplete information and develop a novel approach to cope with the difficulties caused by random delays. To address the problem of information incompleteness, the original measurement sequence is rearranged. Then, we define a new measurement sequence as follows:

1) for $0 \le \mu \le z - \varsigma$,

$$\bar{y}_i^1(\mu) = \begin{bmatrix} y_i^1(\mu) \\ y_i^2(\mu + \varsigma) \end{bmatrix} \tag{13}$$

It is easy to verify that

$$\bar{y}_i^1(\mu) = \bar{\zeta}_1(\mu)\bar{C}_i^1(\mu)x(\mu) + \bar{\zeta}_1(\mu)\bar{D}_i^1(\mu)v_i(\mu) \tag{14}$$

where $\bar{\zeta}_1(\mu)$, $\bar{C}_i^1(\mu)$, and $\bar{D}_i^1(\mu)$ are defined in Appendix A, Eq. (A6).

2) for $\mu \ge z - \varsigma$,

$$\bar{y}_i^2(\mu) = \bar{y}_i^1(\mu) \tag{15}$$

It is easy to verify that

$$\bar{y}_i^2(\mu) = \bar{\zeta}_2(\mu)\bar{C}_i^2(\mu)x(\mu) + \bar{\zeta}_2(\mu)\bar{D}_i^2(\mu)v_i(\mu) \tag{16}$$

where

$$\bar{\zeta}_2(\mu) = \bar{\zeta}_1(\mu), \quad \bar{C}_i^2(\mu) = C_i(\mu), \quad \bar{D}_i^2(\mu) = D_i(\mu) \tag{17}$$

Remark 6: It is important to emphasize that this method rearranges the virtual measurement information, thus achieving effective aggregation of all observation data reflecting the real-time state of the system. This process fully utilizes all available measurement information, thereby overcoming the problem of information incompleteness. The following lemma gives that the generation spaces of the rearranged measurement sequence and the original observation sequence are equivalent.

Lemma 1: The space generated by $\{\{\bar{y}_i^1(\mu)\}_{0\le\mu\le z-\varsigma}, \{\bar{y}_i^2(\mu)\}_{z-\varsigma\le\mu}\}$ is equivalent to the space generated by $\{\{\tilde{y}_i^1(\mu)\}_{0\le\mu\le z-\varsigma}, \{\tilde{y}_i^2(\mu)\}_{z-\varsigma\le\mu}\}$.

Proof. Based on the original virtual measurement information, it is easy to know that $\tilde{y}_i^2(\mu)$ is a linear combination of $\{\{\bar{y}_i^1(\mu)\}_{0\le\mu\le z-\varsigma}, \{\bar{y}_i^2(\mu)\}_{z-\varsigma\le\mu}\}$; one has

$$\tilde{y}_i^2(\mu) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{y}_i^2(\mu) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \bar{y}_i^1(\mu - \varsigma) \tag{18}$$

On the other hand, $\bar{y}_i^1(\mu)$ and $\bar{y}_i^2(\mu)$ are also linear combinations of $\tilde{y}_i^2(\mu)$, as stated in Appendix A, Eq. (A7). Now the proof is completed.

Remark 7: Lemma 1 shows that rearranging measurements and delayed fictitious measurements yield equivalent information spaces, preserving state information. Therefore, projecting the system state onto either generated space uses the same available information set, which guarantees the validity of the subsequent estimator design. It should be emphasized that the rearrangement only changes the order and grouping of the existing virtual measurements. No additional measurement is produced, no observation is repeatedly used, and the original statistical assumptions on the process noise, measurement noise, and transmission variables are not changed. Hence, the rearrangement process does not introduce extra noise correlations or information reuse. Because the rearranged sequence preserves all information in the original measurements while removing the delay terms, the subsequent filter can be designed using the measurement set $\{\{\bar{y}_i^1(\mu)\}_{0\le\mu\le z-\varsigma}, \{\bar{y}_i^2(\mu)\}_{z-\varsigma\le\mu}\}$.

B. THE DYNAMIC VARIABLE PROPERTIES

Before presenting the main results, a basic lemma is given.

Lemma 2: For arbitrary vectors x and y and any scalar $\alpha > 0$, the following relation holds:

$$xy^T + yx^T \preceq \alpha xx^T + \frac{1}{\alpha} yy^T \tag{19}$$

The following are assumptions about the trigger parameters.

Assumption 2: The triggering parameters satisfy the conditions in Appendix A, Eq. (A8).

Then, we present a lemma to show that the dynamic variable is always non-negative.

Lemma 3: Under Assumption 2, for any $(i, j) \in \mathcal{E}$ and $z > 0$, the dynamic variable $\delta_{ij}(z) \geq 0$.

Proof. The proof follows from the preceding norm bounds; due to space limitations, the routine derivation is omitted.

Lemma 4: The tight upper bound of the relative state information $\|\xi_{ij,\mathrm{pri}}(z)\|^2$ is established in Appendix A, Eq. (A9), where $\bar{\delta}^2_{ij}(z)$ is defined in Appendix A, Eq. (A10).

Proof. In light of the ET condition (A4), one has

$$\gamma_{ij}\left(a_{ij}\|\xi_{ij,\mathrm{pri}}(z)\| - \iota_{ij}\right) \leq \delta_{ij}(z) \tag{20}$$

Then, by combining (20) and Lemma 1, we can obtain

$$\|\xi_{ij,\mathrm{pri}}(z)\|^2 \leq \frac{1+\gamma_{ij}}{\gamma_{ij}a^2_{ij}}\left(\delta^2_{ij}(z) + \gamma_{ij}\iota^2_{ij}\right) \tag{21}$$

By using the updated dynamic threshold formula (10), and then combining it with (A9), the bound in Appendix A, Eq. (A11), is obtained. According to (A9) and (A11), one has

$$\|\xi_{ij,\mathrm{pri}}(z)\|^2 \leq \frac{1+\gamma_{ij}}{\gamma_{ij}a^2_{ij}}\left(\bar{\delta}^2_{ij}(z) + \gamma_{ij}\iota^2_{ij}\right) \tag{22}$$

This completes the proof. The upper bound on the relative information established in Lemma 4 is essential for the subsequent derivation of the estimation error covariance matrix.

IV. MAIN RESULTS

A. FILTER DEVELOPMENT

The following multi-step consensus filter is given using the rearranged measurement sequence $\{\{\bar{y}^1_i(\mu)\}_{0\leq\mu\leq z-\varsigma}, \{\bar{y}^2_i(\mu)\}_{z-\varsigma\leq\mu}\}$.

1) for $0 \leq \mu \leq z - \varsigma$,

The corresponding filter is provided in Appendix A as Eq. (A12), where

$$\bar{\varpi}_1(\mu) = \begin{bmatrix} \varpi_1(\mu) & 0 \\ 0 & \varpi_1(\mu+\varsigma)\left(1-\varpi_1(\mu+\varsigma)\right) \end{bmatrix} \tag{23}$$

Here, $\hat{x}^1_{i,\mathrm{pri}}(\mu)$ and $\hat{x}^1_{i,\mathrm{pos}}(\mu)$ denote the prior and posterior estimates of the system state $x(\mu)$, respectively. Moreover, $\hat{x}^{1,i}_{j,\mathrm{pri}}(\mu)$ and $\hat{x}^{1,j}_{i,\mathrm{pri}}(\mu)$ represent the real-time estimates maintained by nodes i and j of their neighboring nodes' prior estimates, respectively. The matrices $K^1_i(\mu)$ and $D^1_{ij}(\mu)$ are consensus estimation gains to be determined. The estimator is initialized by $\hat{x}^1_{i,\mathrm{pos}}(0) = x_0$.

2) for $\mu \geq z - \varsigma$,

The corresponding filter is provided in Appendix A as Eq. (A13), where

$$\bar{\varpi}_2(\mu) = \varpi_2(\mu) \tag{24}$$

Here, $\hat{x}^2_{i,\mathrm{pri}}(\mu)$ and $\hat{x}^2_{i,\mathrm{pos}}(\mu)$ are respectively the prior and posterior estimates of the system state $x(\mu)$. In addition, $\hat{x}^{2,i}_{j,\mathrm{pri}}(\mu)$ and $\hat{x}^{2,j}_{i,\mathrm{pri}}(\mu)$ represent the real-time estimates maintained by nodes i and j of their neighboring nodes' prior estimates, respectively. $K^2_i(\mu)$ and $D^2_{ij}(\mu)$ are the consensus estimation gain parameters to be designed. Based on the multi-step filters, the corresponding prior and posterior errors are defined in Appendix A, Eq. (A14), where $l = 1, 2$. The specific expressions can be represented as follows.

1) for $0 \leq \mu \leq z - \varsigma$,

The corresponding equations are provided in Appendix A as Eqs. (A15) and (A16), where $\xi^1_{ij,\mathrm{pri}}(\mu)$ is defined in Eq. (A17).

2) for $\mu \geq z - \varsigma$,

The corresponding equations are provided in Appendix A as Eqs. (A18) and (A19), where $\xi^2_{ij,\mathrm{pri}}(\mu)$ is defined in Eq. (A20). Building on the derived prior and posterior error dynamics, we next analyze the corresponding second-order error-moment matrices. To this end, the recursive evolution of the system-state second-order moments is first established.

Lemma 5: For the discrete system, $X(\mu)$ satisfies the difference Riccati equation in Appendix A, Eq. (A21), subject to the initial condition $X(0) = X_0$.

Proof. Based on the property of random variable $w(\mu)$, this lemma can be easily proved, and will not be described in detail here.

B. LOCAL MINIMUM VARIANCE CONSENSUS FILTER DESIGN

This section determines the estimation gains by minimizing the error covariance and subsequently formulates the corresponding local minimum-variance filter. The recursive evolution of the system-state second moments is fundamental to establishing the principal results. To simplify notation, we introduce the definitions in Appendix A, Eq. (A22). For any two nodes $j_1, j_2 \in V$, the cross-covariance matrix between the estimation errors is defined in Appendix A, Eq. (A23). To continue, we introduce the following lemma.

Lemma 6: Let W be a real-valued matrix, and let Q denote a random diagonal matrix. Then, the relation in Appendix A, Eq. (A24), holds, where $\circ$ is the Hadamard product.

The following lemmas provide a rigorous mathematical analysis of the evolution of the error covariance.

Lemma 7: Let a_1, a_2, a_3 be given scalars. For the networked system described by (1), the prior and posterior estimation error covariance matrices have upper bounds that evolve according to the following recursions, respectively:

1) for $0 \leq \mu \leq z - \varsigma$,

The corresponding equation is provided in Appendix A as Eq. (A25). where

$$\Phi^1_i(\mu) = I - K^1_i(\mu)\bar{\varpi}_1(\mu)\bar{C}^1_i(\mu) - \sum_{j\in N_i} D^1_{ij}(\mu) \tag{25}$$

with the cross covariance matrix satisfies The corresponding equation is provided in Appendix A as Eq. (A26). and The corresponding equation is provided in Appendix A as Eq. (A27).

2) for $\mu \geq z-\varsigma$,
The corresponding equation is provided in Appendix A as Eq. (A28). where

$$\Phi_i^2(\mu) = I - K_i^2(\mu)\bar{\varpi}_2(\mu)\bar{C}_i^2(\mu) - \sum_{j\in N_i} D_{ij}^2(\mu) \tag{26}$$

with the cross-covariance matrix satisfies
The corresponding equation is provided in Appendix A as Eq. (A29), and the posterior cross-covariance recursion is given in Eq. (A30).
Proof. For simplicity, we only provide the proof for $0 \leq \mu \leq z-\varsigma$. According to (1), the prior covariance is given in Eq. (A31). Then, in light of the posterior error dynamics in (A16), the posterior error consists of the local prior error term, the stochastic transmission term, the measurement noise term, the event-triggering error term, and the neighboring prior error term. To process the coupled terms caused by the event-triggering error and the neighboring prior estimation errors, applying Lemma 2 gives Eq. (A32). Furthermore, by using Eq. (A33) and the relative state bound in (A9), one obtains Eq. (A34).
Therefore, in view of the definition of $\Omega_{i,\text{pos}}^l(\mu)$, one has Eq. (A35). The proof can then be completed using mathematical induction, which would not be elaborated here. For convenience of discussion, we first define the matrices in Appendix A, Eq. (A36). The gain matrices $K_i^l(\mu)$ and $D_{ij}^l(\mu)$ are designed by formulating and solving the following optimization problem:

$$\min_{K_i^l(\mu), D_i^l(\mu)} \operatorname{tr}\left\{\bar{\Omega}_{i,\text{pos}}^l(\mu)\right\} \tag{27}$$

Theorem 1. For networked system (1) subject to random delays and edge-based event-triggered schemes, the consensus gain matrices are derived as follows:
1) for $0 \leq \mu \leq z-\varsigma$,
The corresponding equation is provided in Appendix A as Eq. (A37).
2) for $\mu \geq z-\varsigma$,
The corresponding equation is provided in Appendix A as Eq. (A38), where the auxiliary matrices are defined in Eq. (A39), for $l = 1, 2$. *Proof.* For simplicity, we only provide the proof for $0 \leq \mu \leq z-\varsigma$. Taking the trace on both sides of $\Omega_{i,\text{pos}}^1(\mu)$, one has
The corresponding equation is provided in Appendix A as Eq. (A40). Differentiating with respect to $K_i^1(\mu)$ and equating the result to zero yields Eq. (A41). Thus, it is easy to obtain Eq. (A42). Similarly, differentiating with respect to $D_i^1(\mu)$ and setting the resulting derivative equal to zero yields Eq. (A43). Thus,

$$D_i^1(\mu)\Pi_i^1(\mu) = A_i^1(\mu) - K_i^1(\mu)\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu)A_i^1(\mu) \tag{28}$$

which indicates (A37) hold. Substituting (A37) into (A42), one obtains (A37). This completes the proof.

V. NUMERICAL EXAMPLE

This section provides a detailed numerical example to showcase the efficiency of the proposed filtering method. We analyze both directed and undirected topologies of a connected network with a specified number of 6 nodes. The parameters of system (1) are defined as follows: The corresponding equation is provided in Appendix A as Eq. (A44).

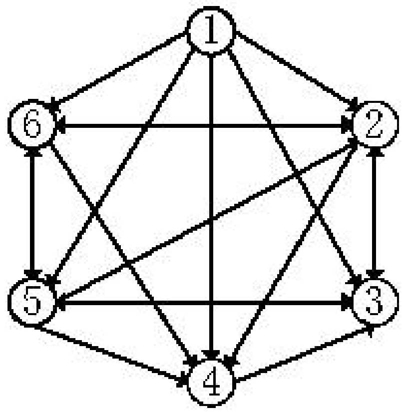

FIGURE 1. Directed Topology Underlying the Network with 6 Nodes

In this simulation, the time horizon is set to $z = 150$. The system topology is shown in Fig. 1. The random variables $w(z)$ and $v_i(z)$ are independent white-noise sequences with zero means and covariance matrices $Q(z)$ and $R_i(z)$, with $R_6(z) = 0.9$. The delay variables are $\varsigma = 25$, $\varpi_1(z) = 0.95$, and $\varpi_2(z) = 0.15$. The trigger parameters are set to $\gamma_{ij} = 2$, $\iota_{ij} = 0.6$, $a_{ij} = 0.1$, $\rho_{ij} = 0.6$, and $\delta_{ij}(0) = 1$. The initial system parameters are

$$x_0 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \qquad P_0 = \begin{bmatrix} 0.5 & 1 \\ 1.5 & 1.8 \end{bmatrix} \tag{29}$$

Figs. 2–5 show the simulation results of the algorithm by utilizing an edge-based dynamic event-triggered mechanism. Specifically, Fig. 2 compares the state of the system with the estimate $\hat{x}^1(z)$, where $x^1(z)$ denotes the first component of $x(z)$. Similarly, Fig. 3 presents the system state and the corresponding estimate $\hat{x}^2(z)$, with $x^2(z)$ representing the second component of $x(z)$. Figs. 4 and 5 represent the corresponding estimation error trajectory. Figs. 2–5 demonstrate that the proposed estimator accurately tracks the moving target, confirming its satisfactory filtering performance.
Fig. 6 shows the triggering instant between node 1 and node 5. Fig. 7 shows the triggering instant between node 1 and node 6; the results for the other nodes are similar.

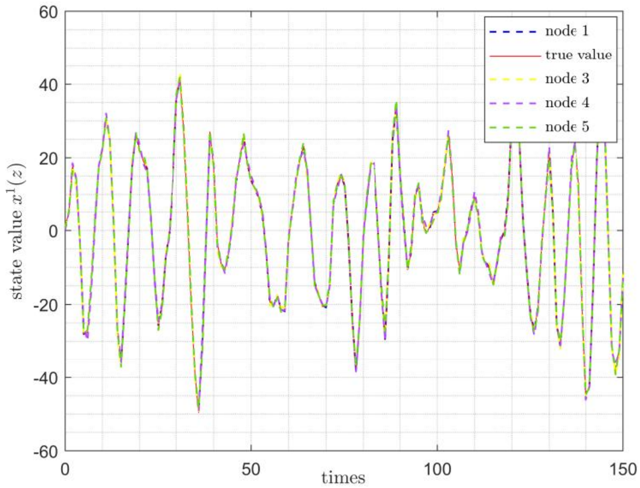

FIGURE 2. The System State and the Estimation of $x^1(z)$

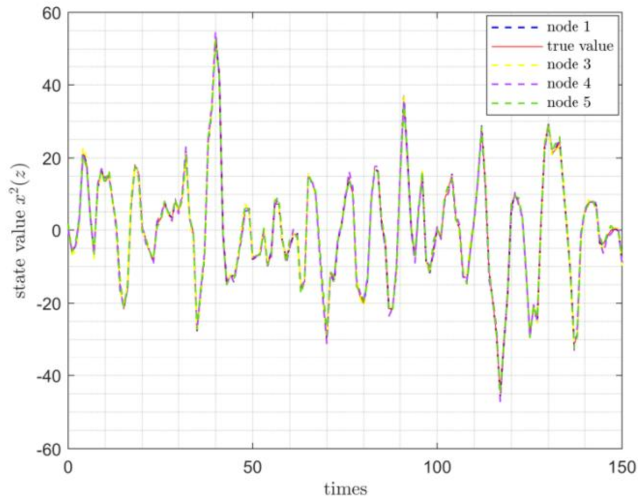

FIGURE 3. The System State and the Estimation of $x^2(z)$

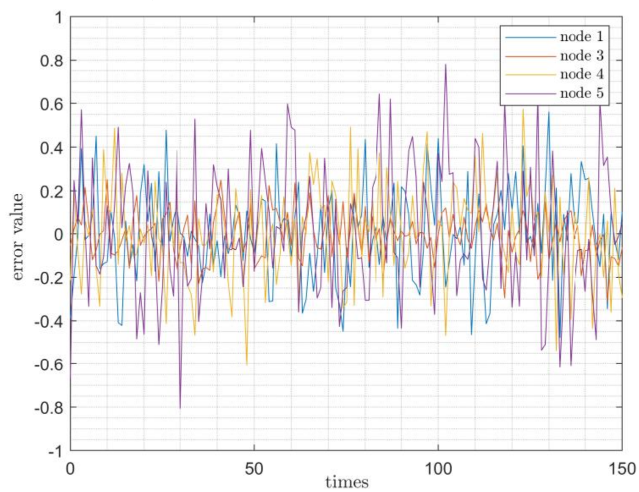

FIGURE 4. The Estimation Error Trajectory

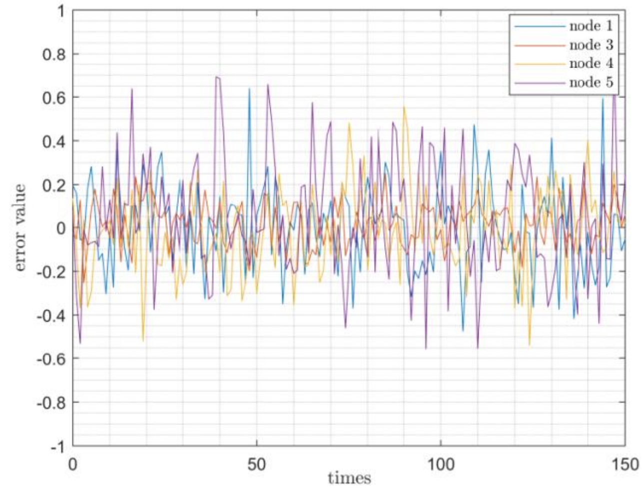

FIGURE 5. The Estimation Error Trajectory

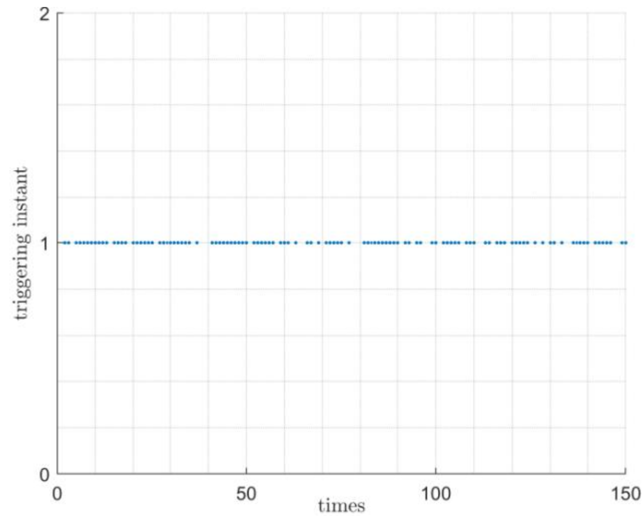

FIGURE 6. The Triggering Instant between Node 1 and Node 5

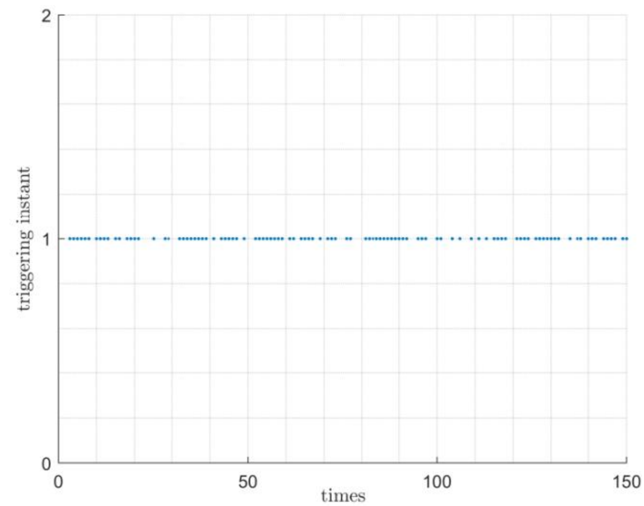

FIGURE 7. The Triggering Instant between Node 1 and Node 6

VI. CONCLUSION

This paper focuses on recursive consensus estimation for a class of networked systems with random transmission delays and an edge-based event-triggering mechanism. A unified probabilistic measurement model has been established by using Bernoulli random variables, which can simultaneously describe delay-free transmission, delayed transmission, and packet dropout. To handle the incomplete measurement information caused by random delays, the delayed observation sequence has been decomposed into virtual measurement sequences and then rearranged into an equivalent delay-free observation sequence. Moreover, a dynamic edge-based sender–receiver event-triggering mechanism is developed to suppress redundant information exchange among neighboring nodes. Based on the rearranged measurement sequence, a multi-step recursive consensus filter has been developed. The prior and posterior estimation-error dynamics are derived, and recursive upper bounds for their covariance matrices are obtained by accounting for stochastic transmission variables and communication errors arising from the edge-based triggering mechanism. Furthermore, by minimizing the trace of the upper bound on the posterior estimation error covariance, the local estimator and consensus gains are determined. The proposed design therefore provides a recursive filtering framework that explicitly incorporates both the rearranged measurement information and the asynchronous communication information among neighboring nodes. It should also be pointed out that the present work is developed under known delay probabilities, known noise statistics, and a fixed communication topology. Future research will expand upon this proposed framework to systems which characterized by unknown or time-varying delay probabilities, switching network topologies, nonlinear dynamics, and joint optimization of triggering parameters and filtering gains.

REFERENCES

[1] X. M. Zhang, Q. L. Han, and X. Yu, "Survey on recent advances in networked control systems," IEEE Trans. Ind. Informat., vol. 12, no. 11, pp. 1740–1752, 2016.

[2] Y. Chen, Z. Wang, L. Wang, and W. Sheng, "Finite-horizon H∞state estimation for stochastic coupled networks with random inner couplings using round-robin protocol," IEEE Trans. Cybern., vol. 51, no. 5, pp. 1204–1215, 2021.

[3] Z. Yang, L. Yu, D. Liu, and W. Zhang, "Grouped-round-robin-based state estimation for multiplex networks subject to deception attacks," ISA Trans., vol. 169, pp. 329–341, 2025.

[4] Y. Chen, C. Han, W. Wang, and J. Xu, "Outlier-resistant filtering for networked 2-D FMLSS systems with dead-zone-like censoring under asynchronous random delays," IEEE Trans. Autom. Sci. Eng., vol. 23, pp. 4797–4818, 2026.

[5] S. Chen and D. W. C. Ho, "Edge-based sender-receiver event-triggered schemes for distributed filtering," IEEE Trans. Circuits Syst. I, Reg. Papers, vol. 70, no. 11, pp. 2143–2155, 2023.

[6] X. Liu, C. Han, and W. Wang, "Consensus optimal filter for linear systems over amplify-and-forward relay networks: Handling delayed and degraded channels," IEEE Trans. Netw. Sci. Eng., vol. 11, no. 11, pp. 4076–4092, 2024.

[7] H. Ren, Z. Cheng, J. Qin, and R. Lu, "Deception attacks on event-triggered distributed consensus estimation for nonlinear systems," Automatica, vol. 154, p. 111100, 2023.

[8] X. Ge, Q. L. Han, X. M. Zhang, L. Ding, and F. Yang, "Distributed event-triggered estimation over sensor networks: A survey," IEEE Trans. Cybern., vol. 50, no. 5, pp. 1306–1320, 2020.

[9] R. Caballero-Aguila, A. Hermoso-Carazo, and J. Linares-Perez, "Networked distributed fusion estimation under uncertain outputs with random transmission delays, packet losses and multi-packet processing," Signal Process., vol. 156, pp. 71–83, 2019.

[10] Q. Li, Z. Wang, B. Shen, H. Liu, and W. Sheng, "A resilient approach to recursive distributed filtering for multirate systems over sensor networks with time-correlated fading channels," IEEE Trans. Signal Inf. Process. Netw., vol. 7, pp. 636–647, 2021.

[11] X. Meng, Z. Wang, F. Wang, and Y. Chen, "Distributed fusion filtering for nonlinear time-varying systems over amplify-and-forward relay networks: A quantized framework," IEEE Trans. Netw. Sci. Eng., vol. 10, no. 6, pp. 3597–3608, 2023.

[12] L. Wang, Z. Wang, B. Shen, and G. Wei, "Recursive filtering with measurement fading: A multiple description coding scheme," IEEE Trans. Autom. Control, vol. 66, no. 23, pp. 5144–5159, 2021.

[13] W. Michiels, "To filter or not to filter? Impact on stability of delay-difference and neutral equations with multiple delays," IEEE Trans. Autom. Control, vol. 68, no. 12, pp. 3687–3693, 2023.

[14] Y. Chen, Z. Wang, and Y. Liu, "Privacy-preserving pinning synchronization for time-delay complex networks: A noise injection scheme," in Proc. 30th Int. Conf. Autom. Comput. (ICAC), Loughborough, United Kingdom, pp. 1–6, 2025.

[15] M. S. Aslam, H. Bilal, W. J. Chang, N. Kumar, I. A. Khan, and A. V. Vasilakos, "Delayed filtering of Markov jump fuzzy systems in consumer electronics: Input-output analysis," IEEE Trans. Consum. Electron., vol. 71, no. 2, pp. 7002–7013, 2025.

[16] J. Hu, C. Wang, R. Caballero-Aguila, and H. Liu, "Distributed optimal fusion filtering for singular systems with random transmission delays and packet dropout compensations," Commun. Nonlinear Sci. Numer. Simul., vol. 119, p. 107093, 2023.

[17] X. M. Zhang, Q. L. Han, and X. Ge, "A novel approach to performance analysis of discrete-time networked systems subject to network-induced delays and malicious packet dropouts," Automatica, vol. 136, p. 110010, 2022.

[18] Y. Pan, Y. Wu, and H. K. Lam, "Security-based fuzzy control for nonlinear networked control systems with DoS attacks via a resilient event-triggered scheme," IEEE Trans. Fuzzy Syst., vol. 30, no. 20, pp. 4359–4368, 2022.

[19] X. Ge, Q. L. Han, and Z. Wang, "A dynamic event-triggered transmission scheme for distributed set-membership estimation over wireless sensor networks," IEEE Trans. Cybern., vol. 49, no. 1, pp. 171–183, 2019.

[20] D. Ding, Z. Wang, and Q. L. Han, "A set-membership approach to event-triggered filtering for general nonlinear systems over sensor networks," IEEE Trans. Autom. Control, vol. 65, no. 10, pp. 1792–1799, 2020.

[21] Z. Li, Y. Liu, X. Hu, and W. Dai, "Event-triggered optimal Kalman consensus filter with upper bound of error covariance," Signal Process., vol. 188, p. 108175, 2021.

[22] C. Nowzari, E. Garcia, and J. Cortes, "Event-triggered communication and control of networked systems for multi-agent consensus," Automatica, vol. 105, pp. 1–27, 2019.

[23] Q. Li, Z. Wang, J. Hu, and W. Sheng, "Distributed state and fault estimation over sensor networks with probabilistic quantizations: The dynamic event-triggered case," Automatica, vol. 131, p. 109784, 2021.

[24] L. Xu, Y. Mo, and L. Xie, "Remote state estimation with stochastic event-triggered sensor schedule and packet drops," IEEE Trans. Autom. Control, vol. 65, no. 23, pp. 4981–4988, 2020.

[25] S. Guo and X. Meng, "Fully distributed control of multiagent networks with edge-based event-triggered communication," IEEE Trans. Autom. Control, vol. 68, no. 15, pp. 5630–5637, 2023.

[26] H. Zhao, X. Meng, and S. Wu, "Distributed edge-based event-triggered coordination control for multi-agent systems," Automatica, vol. 132, p. 109797, 2021.

APPENDIX A. EQUATIONS EXCEEDING ONE-THIRD PAGE WIDTH

The following equations are moved from the main text because their actual displayed formula width exceeds one-third of the usable page width. Their mathematical content is unchanged.

$$(\zeta_1(z),\zeta_2(z)) = \begin{cases}(1,\cdot), & \text{delay-free transmission,}\\ (0,1), & \text{delayed transmission,}\\ (0,0), & \text{packet dropout.}\end{cases} \tag{A1}$$

$$\tilde{x}^{j}_{i,\text{pri}}(z) = \begin{cases}\hat{x}_{i,\text{pri}}(z), & \text{if edge } (i,j) \text{ is triggered,}\\ \hat{x}^{j}_{i,\text{pri}}(z), & \text{if edge } (i,j) \text{ is not triggered.}\end{cases} \tag{A2}$$

$$\begin{aligned}\xi_{ij,\text{pri}}(z) &= e^{j}_{i,\text{pri}}(z) - e^{i}_{j,\text{pri}}(z)\\ &= \big(\hat{x}^{j}_{i,\text{pri}}(z) - \hat{x}_{i,\text{pri}}(z)\big) - \big(\hat{x}^{i}_{j,\text{pri}}(z) - \hat{x}_{j,\text{pri}}(z)\big)\end{aligned} \tag{A3}$$

$$z^{i\to j}_{s+1} = \inf_{t>z^{i\to j}_s}\left\{t : \mathcal{F}\big(\xi_{ij,\text{pri}}(z)\big) > \delta_{ij}(z),\ \forall z\in(z^{i\to j}_s,t]\right\} \tag{A4}$$

$$\begin{cases}y^1_i(z) = \zeta_1(z)C_i(z)x(z) + \zeta_1(z)D_i(z)v_i(z),\\ y^2_i(z) = \zeta_2(z)(1-\zeta_1(z))C_i(z-\varsigma)x(z-\varsigma) + \zeta_2(z)(1-\zeta_1(z))D_i(z-\varsigma)v_i(z-\varsigma)\end{cases} \tag{A5}$$

$$\bar{\zeta}_1(\mu) = \begin{bmatrix}\zeta_1(\mu) & 0\\ 0 & \zeta_2(\mu+\varsigma)(1-\zeta_2(\mu+\varsigma))\end{bmatrix},\quad \bar{C}^1_i(\mu) = \begin{bmatrix}C_i(\mu)\\ C_i(\mu)\end{bmatrix},\quad \bar{D}^1_i(\mu) = \begin{bmatrix}D_i(\mu)\\ D_i(\mu)\end{bmatrix} \tag{A6}$$

$$\begin{aligned}\bar{y}^1_i(\mu) &= \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\tilde{y}^2_i(\mu) + \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\tilde{y}^2_i(\mu+\varsigma),\\ \bar{y}^2_i(\mu) &= \begin{bmatrix}1 & 0\end{bmatrix}\tilde{y}^2_i(\mu)\end{aligned} \tag{A7}$$

$$0<\rho_{ij}<1,\ \gamma_{ij}\rho_{ij}\ge 1,\ \gamma_{ij}=\gamma_{ji},\ \iota_{ij}=\iota_{ji},\ a_{ij}=a_{ji},\ \delta_{ij}(0)=\delta_{ji}(0) \tag{A8}$$

$$\|\xi_{ij,\text{pri}}(z)\|^2 \le \Gamma_{ij,\text{pri}}(z) \triangleq \frac{1+\gamma_{ij}}{\gamma_{ij}a^2_{ij}}\big(\bar{\delta}^2_{ij}(z) + \gamma_{ij}\iota^2_{ij}\big) \tag{A9}$$

$$\bar{\delta}^2_{ij}(z) = \left(2\rho^2_{ij} + \frac{1+\gamma_{ij}}{\gamma^2_{ij}}\right)^{z}\delta^2_{ij}(0) + \left(\frac{1+3\gamma_{ij}}{\gamma_{ij}}\right)\iota^2_{ij}\sum_{n=1}^{z-1}\left(2\rho^2_{ij} + \frac{1+\gamma_{ij}}{\gamma^2_{ij}}\right)^{n} \tag{A10}$$

$$\begin{aligned}\delta^2_{ij}(z+1) &\le 2\rho^2_{ij}\delta^2_{ij}(z) + 2\iota^2_{ij} + a^2_{ij}\|\xi_{ij,\text{pri}}(z)\|^2\\ &\le \left(2\rho^2_{ij} + \frac{1+\gamma_{ij}}{\gamma^2_{ij}}\right)\delta^2_{ij}(z) + \left(\frac{1+3\gamma_{ij}}{\gamma_{ij}}\right)\iota^2_{ij}\\ &\le \cdots\\ &\le \left(2\rho^2_{ij} + \frac{1+\gamma_{ij}}{\gamma^2_{ij}}\right)^{z+1}\delta^2_{ij}(0) + \left(\frac{1+3\gamma_{ij}}{\gamma_{ij}}\right)\iota^2_{ij}\sum_{n=1}^{z}\left(2\rho^2_{ij} + \frac{1+\gamma_{ij}}{\gamma^2_{ij}}\right)^{n}\end{aligned} \tag{A11}$$

$$\begin{aligned}\text{Priori}\quad \hat{x}^1_{i,\text{pri}}(\mu) &= A(\mu-1)\hat{x}^1_{i,\text{pos}}(\mu-1),\\ \text{Posterior}\quad \hat{x}^1_{i,\text{pos}}(\mu) &= \hat{x}^1_{i,\text{pri}}(\mu) + K^1_i(\mu)\big(\bar{y}^1_i(\mu) - \bar{\varpi}_1(\mu)\bar{C}^1_i(\mu)\hat{x}^1_{i,\text{pri}}(\mu)\big)\\ &\quad + \sum_{j\in N_i} D^1_{ij}(\mu)\big(\hat{x}^{1,i}_{j,\text{pri}}(\mu) - \hat{x}^{1,j}_{i,\text{pri}}(\mu)\big)\end{aligned} \tag{A12}$$

$$\begin{aligned}\text{Priori}\quad \hat{x}^2_{i,\text{pri}}(\mu) &= A(\mu-1)\hat{x}^2_{i,\text{pos}}(\mu-1),\\ \text{Posterior}\quad \hat{x}^2_{i,\text{pos}}(\mu) &= \hat{x}^2_{i,\text{pri}}(\mu) + K^2_i(\mu)\big(\bar{y}^2_i(\mu) - \bar{\varpi}_2(\mu)\bar{C}^2_i(\mu)\hat{x}^2_{i,\text{pri}}(\mu)\big)\\ &\quad + \sum_{j\in N_i} D^2_{ij}(\mu)\big(\hat{x}^{2,i}_{j,\text{pri}}(\mu) - \hat{x}^{2,j}_{i,\text{pri}}(\mu)\big)\end{aligned} \tag{A13}$$

$$e^l_{i,\text{pri}}(\mu) \triangleq x(\mu) - \hat{x}^l_{i,\text{pri}}(\mu),\qquad e^l_{i,\text{pos}}(\mu) \triangleq x(\mu) - \hat{x}^l_{i,\text{pos}}(\mu) \tag{A14}$$

$$e^1_{i,\text{pri}}(\mu) = A(\mu-1)e^1_{i,\text{pos}}(\mu-1) + B(\mu-1)w(\mu-1) \tag{A15}$$

$$\begin{aligned} e^1_{i,\mathrm{pos}}(\mu) &= \left(I - K^1_i(\mu)\varpi_1(\mu)\bar{C}^1_i(\mu) - \sum_{j\in N_i} D^1_{ij}(\mu)\right) e^1_{i,\mathrm{pri}}(\mu) \\ &\quad - K^1_i(\mu)\big(\bar{\zeta}_1(\mu) - \bar{\varpi}_1(\mu)\big)\bar{C}^1_i(\mu)x(\mu) - K^1_i(\mu)\bar{\zeta}_1(\mu)\bar{D}^1_i(\mu)v_i(\mu) \\ &\quad + \sum_{j\in N_i} D^1_{ij}(\mu)\xi^1_{ij,\mathrm{pri}}(\mu) + \sum_{j\in N_i} D^1_{ij}(\mu)e^1_{j,\mathrm{pri}}(\mu) \end{aligned} \tag{A16}$$

$$\begin{aligned} \xi^1_{ij,\mathrm{pri}}(\mu) &= e^{1,j}_{i,\mathrm{pri}}(\mu) - e^{1,i}_{j,\mathrm{pri}}(\mu) \\ &= \big(\hat{x}^{1,j}_{i,\mathrm{pri}}(\mu) - \hat{x}^1_{i,\mathrm{pri}}(\mu)\big) - \big(\hat{x}^{1,i}_{j,\mathrm{pri}}(\mu) - \hat{x}^1_{j,\mathrm{pri}}(\mu)\big) \end{aligned} \tag{A17}$$

$$e^2_{i,\mathrm{pri}}(\mu) = A(\mu-1)e^2_{i,\mathrm{pos}}(\mu-1) + B(\mu-1)w(\mu-1) \tag{A18}$$

$$\begin{aligned} e^2_{i,\mathrm{pos}}(\mu) &= \left(I - K^2_i(\mu)\varpi_2(\mu)\bar{C}^2_i(\mu) - \sum_{j\in N_i} D^2_{ij}(\mu)\right) e^2_{i,\mathrm{pri}}(\mu) \\ &\quad - K^2_i(\mu)\big(\bar{\zeta}_2(\mu) - \bar{\varpi}_2(\mu)\big)\bar{C}^2_i(\mu)x(\mu) - K^2_i(\mu)\bar{\zeta}_2(\mu)\bar{D}^2_i(\mu)v_i(\mu) \\ &\quad + \sum_{j\in N_i} D^2_{ij}(\mu)\xi^2_{ij,\mathrm{pri}}(\mu) + \sum_{j\in N_i} D^2_{ij}(\mu)e^2_{j,\mathrm{pri}}(\mu) \end{aligned} \tag{A19}$$

$$\begin{aligned} \xi^2_{ij,\mathrm{pri}}(\mu) &= e^{2,j}_{i,\mathrm{pri}}(\mu) - e^{2,i}_{j,\mathrm{pri}}(\mu) \\ &= \big(\hat{x}^{2,j}_{i,\mathrm{pri}}(\mu) - \hat{x}^2_{i,\mathrm{pri}}(\mu)\big) - \big(\hat{x}^{2,i}_{j,\mathrm{pri}}(\mu) - \hat{x}^2_{j,\mathrm{pri}}(\mu)\big) \end{aligned} \tag{A20}$$

$$X(\mu) = A(\mu-1)X(\mu-1)A^T(\mu-1) + B(\mu-1)Q(\mu-1)B^T(\mu-1) \tag{A21}$$

$$P^l_{i,\mathrm{pri}}(\mu) \triangleq \hat{\mathbb{E}}\{e^l_{i,\mathrm{pri}}(e^l_{i,\mathrm{pri}})^T\}, \quad P^l_{i,\mathrm{pos}}(\mu) \triangleq \hat{\mathbb{E}}\{e^l_{i,\mathrm{pos}}(e^l_{i,\mathrm{pos}})^T\} \tag{A22}$$

$$\Omega^l_{\mathrm{pri},j_1j_2}(\mu) \triangleq \hat{\mathbb{E}}\{e^l_{j_1,\mathrm{pri}}(\mu)(e^l_{j_2,\mathrm{pri}}(\mu))^T\}, \quad \Omega^l_{\mathrm{pos},j_1j_2}(\mu) \triangleq \hat{\mathbb{E}}\{e^l_{j_1,\mathrm{pos}}(\mu)(e^l_{j_2,\mathrm{pos}}(\mu))^T\} \tag{A23}$$

$$\hat{\mathbb{E}}\{W^TQW\} = \begin{bmatrix} \hat{\mathbb{E}}\{w_1^2\} & \hat{\mathbb{E}}\{w_1w_2\} & \cdots & \hat{\mathbb{E}}\{w_1w_m\} \\ \hat{\mathbb{E}}\{w_2w_1\} & \hat{\mathbb{E}}\{w_2^2\} & \cdots & \hat{\mathbb{E}}\{w_2w_m\} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\mathbb{E}}\{w_mw_1\} & \hat{\mathbb{E}}\{w_mw_2\} & \cdots & \hat{\mathbb{E}}\{w_m^2\} \end{bmatrix} \circ Q \tag{A24}$$

$$\begin{aligned} \Omega^1_{i,\mathrm{pri}}(\mu) &= A(\mu-1)\Omega^1_{i,\mathrm{pos}}(\mu-1)A^T(\mu-1) + B(\mu-1)Q(\mu-1)B^T(\mu-1), \\ \Omega^1_{i,\mathrm{pos}}(\mu) &= (1+a_1)\Phi^1_i(\mu)\Omega^1_{i,\mathrm{pri}}(\mu)(\Phi^1_i(\mu))^T \\ &\quad + K^1_i(\mu)\{(I-\bar{\varpi}_1(\mu))\bar{\varpi}_1(\mu)\circ(\bar{C}^1_i(\mu)X(\mu)(\bar{C}^1_i(\mu))^T) + \bar{\varpi}_1(\mu)\circ\bar{R}^1_i(\mu)\}(K^1_i(\mu))^T \\ &\quad + (1+a_1^{-1}+a_2^{-1})n_i\sum_{j\in N_i} D^1_{ij}(\mu)\Gamma^1_{ij,\mathrm{pri}}(\mu)(D^1_{ij}(\mu))^T \\ &\quad + (1+a_2)\sum_{j_1\in N_i}\sum_{j_2\in N_i} D^1_{ij_1}(\mu)\Omega^1_{j_1j_2,\mathrm{pri}}(\mu)(D^1_{ij_2}(\mu))^T \\ &\quad + \Phi^1_i(\mu)\sum_{j\in N_i}\Omega^1_{ij,\mathrm{pri}}(\mu)(D^1_{ij}(\mu))^T + \sum_{j\in N_i} D^1_{ij}(\mu)(\Omega^1_{ij,\mathrm{pri}}(\mu))^T(\Phi^1_i(\mu))^T \end{aligned} \tag{A25}$$

$$\Omega^1_{j_1j_2,\mathrm{pri}}(\mu) = A(\mu-1)\Omega^1_{j_1j_2,\mathrm{pos}}(\mu-1)A^T(\mu-1) + B(\mu-1)Q(\mu-1)B^T(\mu-1) \tag{A26}$$

$$\begin{aligned} \Omega^1_{j_1j_2,\mathrm{pos}}(\mu) &= \Phi^1_{j_1}(\mu)\Omega^1_{j_1j_2,\mathrm{pri}}(\mu)(\Phi^1_{j_2}(\mu))^T + \Phi^1_{j_1}(\mu)\sum_{h\in N_{j_2}}\Omega^1_{j_1h,\mathrm{pri}}(\mu)(D^1_{hj_2}(\mu))^T \\ &\quad + \sum_{m\in N_{j_1}} D^1_{mj_1}(\mu)\Omega^1_{mj_2,\mathrm{pri}}(\mu)(\Phi^1_{j_1}(\mu))^T + \sum_{m\in N_{j_1}}\sum_{h\in N_{j_2}} D^1_{mj_1}(\mu)\Omega^1_{mh,\mathrm{pri}}(\mu)(D^1_{hj_2}(\mu))^T \\ &\quad + a_3n_{j_1}\sum_{m\in N_{j_1}} D^1_{mj_1}(\mu)\Gamma^1_{mj_1,\mathrm{pri}}(\mu)(D^1_{mj_1}(\mu))^T + a_3^{-1}n_{j_2}\sum_{h\in N_{j_2}} D^1_{hj_2}(\mu)\Gamma^1_{hj_2,\mathrm{pri}}(\mu)(D^1_{hj_2}(\mu))^T \end{aligned} \tag{A27}$$

$$\begin{aligned}\Omega^2_{i,\text{pri}}(\mu) &= A(\mu-1)\Omega^2_{i,\text{pos}}(\mu-1)A^T(\mu-1)+B(\mu-1)Q(\mu-1)B^T(\mu-1),\\ \Omega^2_{i,\text{pos}}(\mu) &= (1+a_1)\Phi^2_i(\mu)\Omega^2_{i,\text{pri}}(\mu)(\Phi^2_i(\mu))^T\\ &\quad + K^2_i(\mu)\{(I-\bar{\varpi}_2(\mu))\bar{\varpi}_2(\mu)\circ(\bar{C}^2_i(\mu)X(\mu)(\bar{C}^2_i(\mu))^T)+\bar{\varpi}_2(\mu)\circ\bar{R}^2_i(\mu)\}(K^2_i(\mu))^T\\ &\quad + (1+a_1^{-1}+a_2^{-1})n_i\sum_{j\in N_i}D^2_{ij}(\mu)\Gamma^2_{ij,\text{pri}}(\mu)(D^2_{ij}(\mu))^T\\ &\quad + (1+a_2)\sum_{j_1\in N_i}\sum_{j_2\in N_i}D^2_{ij_1}(\mu)\Omega^2_{j_1j_2,\text{pri}}(\mu)(D^2_{ij_2}(\mu))^T\\ &\quad + \Phi^2_i(\mu)\sum_{j\in N_i}\Omega^2_{ij,\text{pri}}(\mu)(D^2_{ij}(\mu))^T+\sum_{j\in N_i}D^2_{ij}(\mu)(\Omega^2_{ij,\text{pri}}(\mu))^T(\Phi^2_i(\mu))^T\end{aligned} \tag{A28}$$

$$\Omega^2_{j_1j_2,\text{pri}}(\mu) = A(\mu-1)\Omega^2_{j_1j_2,\text{pos}}(\mu-1)A^T(\mu-1)+B(\mu-1)Q(\mu-1)B^T(\mu-1) \tag{A29}$$

$$\begin{aligned}\Omega^2_{j_1j_2,\text{pos}}(\mu) &= \Phi^2_{j_1}(\mu)\Omega^2_{j_1j_2,\text{pri}}(\mu)(\Phi^2_{j_2}(\mu))^T+\Phi^2_{j_1}(\mu)\sum_{h\in N_{j_2}}\Omega^2_{j_1h,\text{pri}}(\mu)(D^2_{hj_2}(\mu))^T\\ &\quad + \sum_{m\in N_{j_1}}D^2_{mj_1}(\mu)\Omega^2_{mj_2,\text{pri}}(\mu)(\Phi^2_{j_1}(\mu))^T+\sum_{m\in N_{j_1}}\sum_{h\in N_{j_2}}D^2_{mj_1}(\mu)\Omega^2_{mh,\text{pri}}(\mu)(D^2_{hj_2}(\mu))^T\\ &\quad + a_3n_{j_1}\sum_{m\in N_{j_1}}D^2_{mj_1}(\mu)\Gamma^2_{mj_1,\text{pri}}(\mu)(D^2_{mj_1}(\mu))^T+a_3^{-1}n_{j_2}\sum_{h\in N_{j_2}}D^2_{hj_2}(\mu)\Gamma^2_{hj_2,\text{pri}}(\mu)(D^2_{hj_2}(\mu))^T\end{aligned} \tag{A30}$$

$$P^1_{i,\text{pri}}(\mu) = A(\mu-1)P^1_{i,\text{pos}}(\mu-1)A^T(\mu-1)+B(\mu-1)Q(\mu-1)B^T(\mu-1) \tag{A31}$$

$$\begin{aligned}&\left(\Phi^1_i(\mu)e^1_{i,\text{pri}}(\mu)+\sum_{j\in N_i}D^1_{ij}(\mu)\xi^1_{ij,\text{pri}}(\mu)+\sum_{j\in N_i}D^1_{ij}(\mu)e^1_{j,\text{pri}}(\mu)\right)\\ &\times\left(\Phi^1_i(\mu)e^1_{i,\text{pri}}(\mu)+\sum_{j\in N_i}D^1_{ij}(\mu)\xi^1_{ij,\text{pri}}(\mu)+\sum_{j\in N_i}D^1_{ij}(\mu)e^1_{j,\text{pri}}(\mu)\right)^T\\ &\le (1+a_1)\Phi^1_i(\mu)e^1_{i,\text{pri}}(\mu)(e^1_{i,\text{pri}}(\mu))^T(\Phi^1_i(\mu))^T\\ &+(1+a_1^{-1}+a_2^{-1})\left(\sum_{j\in N_i}D^1_{ij}(\mu)\xi^1_{ij,\text{pri}}(\mu)\right)\left(\sum_{j\in N_i}D^1_{ij}(\mu)\xi^1_{ij,\text{pri}}(\mu)\right)^T\\ &+(1+a_2)\left(\sum_{j\in N_i}D^1_{ij}(\mu)e^1_{j,\text{pri}}(\mu)\right)\left(\sum_{j\in N_i}D^1_{ij}(\mu)e^1_{j,\text{pri}}(\mu)\right)^T\\ &+\Phi^1_i(\mu)e^1_{i,\text{pri}}(\mu)\left(\sum_{j\in N_i}D^1_{ij}(\mu)e^l_{j,\text{pri}}(\mu)\right)^T+\left(\sum_{j\in N_i}D^1_{ij}(\mu)e^1_{j,\text{pri}}(\mu)\right)(e^1_{i,\text{pri}}(\mu))^T(\Phi^1_i(\mu))^T\end{aligned} \tag{A32}$$

$$\left(\sum_{j\in N_i}D^1_{ij}(\mu)\xi^1_{ij,\text{pri}}(\mu)\right)\left(\sum_{j\in N_i}D^1_{ij}(\mu)\xi^1_{ij,\text{pri}}(\mu)\right)^T\le n_i\sum_{j\in N_i}D^1_{ij}(\mu)\xi^1_{ij,\text{pri}}(\mu)(\xi^1_{ij,\text{pri}}(\mu))^T(D^1_{ij}(\mu))^T \tag{A33}$$

$$\left(\sum_{j\in N_i}D^1_{ij}(\mu)\xi^1_{ij,\text{pri}}(\mu)\right)\left(\sum_{j\in N_i}D^1_{ij}(\mu)\xi^1_{ij,\text{pri}}(\mu)\right)^T\le n_i\sum_{j\in N_i}D^1_{ij}(\mu)\Gamma_{ij,\text{pri}}(\mu)(D^1_{ij}(\mu))^T \tag{A34}$$

$$\begin{aligned}P^1_{i,\text{pos}}(\mu) &\le (1+a_1)\Phi^1_i(\mu)P^1_{i,\text{pri}}(\mu)(\Phi^1_i(\mu))^T\\ &\quad + K^1_i(\mu)\{(I-\bar{\varpi}_1(\mu))\bar{\varpi}_1(\mu)\circ(\bar{C}^1_i(\mu)X(\mu)(\bar{C}^1_i(\mu))^T)+\bar{\varpi}_1(\mu)\circ\bar{R}^1_i(\mu)\}(K^1_i(\mu))^T\\ &\quad + (1+a_1^{-1}+a_2^{-1})n_i\sum_{j\in N_i}D^1_{ij}(\mu)\Gamma^1_{ij,\text{pri}}(\mu)(D^1_{ij}(\mu))^T\\ &\quad + (1+a_2)\sum_{j_1\in N_i}\sum_{j_2\in N_i}D^1_{ij_1}(\mu)P^1_{j_1j_2,\text{pri}}(\mu)(D^1_{ij_2}(\mu))^T\\ &\quad + \Phi^1_i(\mu)\sum_{j\in N_i}P^1_{ij,\text{pri}}(\mu)(D^1_{ij}(\mu))^T+\sum_{j\in N_i}D^1_{ij}(\mu)(P^1_{ij,\text{pri}}(\mu))^T(\Phi^1_i(\mu))^T\end{aligned} \tag{A35}$$

$$
\begin{aligned}
D_i^l(\mu) &\triangleq [D_{ij_1}^l(\mu)\ D_{ij_2}^l(\mu)\ \cdots\ D_{ij_{n_i}}^l(\mu)],\\
\varepsilon_i &\triangleq [I\ I\ \cdots\ I]^T,\\
G_i^l(\mu) &\triangleq [\Omega_{ij_1,\mathrm{pri}}^l(\mu)\ \Omega_{ij_2,\mathrm{pri}}^l(\mu)\ \cdots\ \Omega_{ij_{n_i},\mathrm{pri}}^l(\mu)],\\
U_i^l(\mu) &\triangleq \begin{bmatrix} \Omega_{j_1j_1,\mathrm{pri}}^l & \Omega_{j_1j_2,\mathrm{pri}}^l & \cdots & \Omega_{j_1j_{n_i},\mathrm{pri}}^l \\ \Omega_{j_2j_1,\mathrm{pri}}^l & \Omega_{j_2j_2,\mathrm{pri}}^l & \cdots & \Omega_{j_2j_{n_i},\mathrm{pri}}^l \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{j_{n_i}j_1,\mathrm{pri}}^l & \Omega_{j_{n_i}j_2,\mathrm{pri}}^l & \cdots & \Omega_{j_{n_i}j_{n_i},\mathrm{pri}}^l \end{bmatrix},\\
\Gamma_i^l(\mu) &\triangleq \mathrm{diag}\{\Gamma_{ij_1,\mathrm{pri}}(\mu), \Gamma_{ij_2,\mathrm{pri}}(\mu), \ldots, \Gamma_{ij_{n_i},\mathrm{pri}}(\mu)\}
\end{aligned}
\tag{A36}
$$

$$
\begin{aligned}
K_i^1(\mu) &= \left[\Omega_{i,\mathrm{pri}}^1(\mu)(\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu))^T - A_i^1(\mu)(\Pi_i^1(\mu))^{-1}(A_i^1(\mu))^T(\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu))^T\right](\Lambda_i^1(\mu))^{-1},\\
D_i^1(\mu) &= \left[A_i^1(\mu) - K_i^1(\mu)\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu)A_i^1(\mu)\right](\Pi_i^1(\mu))^{-1}
\end{aligned}
\tag{A37}
$$

$$
\begin{aligned}
K_i^2(\mu) &= \left[\Omega_{i,\mathrm{pri}}^2(\mu)(\bar{\varpi}_2(\mu)\bar{C}_i^2(\mu))^T - A_i^2(\mu)(\Pi_i^2(\mu))^{-1}(A_i^2(\mu))^T(\bar{\varpi}_2(\mu)\bar{C}_i^2(\mu))^T\right](\Lambda_i^2(\mu))^{-1},\\
D_i^2(\mu) &= \left[A_i^2(\mu) - K_i^2(\mu)\bar{\varpi}_2(\mu)\bar{C}_i^2(\mu)A_i^2(\mu)\right](\Pi_i^2(\mu))^{-1}
\end{aligned}
\tag{A38}
$$

$$
\begin{aligned}
\Psi_i^l(\mu) &\triangleq \bar{\varpi}_l(\mu)\bar{C}_i^l(\mu)\Omega_{i,\mathrm{pri}}^l(\mu)(\bar{\varpi}_l(\mu)\bar{C}_i^l(\mu))^T\\
&\quad + (I - \bar{\varpi}_l(\mu))\bar{\varpi}_l(\mu)\circ(\bar{C}_i^l(\mu)X(\mu)(\bar{C}_i^l(\mu))^T) + \bar{\varpi}_l(\mu)\circ\bar{R}_i^l(\mu),\\
A_i^l(\mu) &\triangleq \Omega_{i,\mathrm{pri}}^l(\mu)\varepsilon_i^T - G_i^l(\mu),\\
\Pi_i^l(\mu) &\triangleq \varepsilon_i\Omega_{i,\mathrm{pri}}^l(\mu)\varepsilon_i^T + U_i^l(\mu) - \varepsilon_iG_i^l(\mu) - (G_i^l(\mu))^T\varepsilon_i^T,\\
\Lambda_i^l(\mu) &\triangleq \Psi_i^l(\mu) - \bar{\varpi}_l(\mu)\bar{C}_i^l(\mu)A_i^l(\mu)(\Pi_i^l(\mu))^{-1}(A_i^l(\mu))^T(\bar{\varpi}_l(\mu)\bar{C}_i^l(\mu))^T
\end{aligned}
\tag{A39}
$$

$$
\begin{aligned}
\mathrm{tr}\{\Omega_{i,\mathrm{pos}}^1(\mu)\} &= \mathrm{tr}\{\Omega_{i,\mathrm{pri}}^1(\mu)\} - 2\,\mathrm{tr}\{\Omega_{i,\mathrm{pri}}^1(\mu)\varepsilon_i^T(D_i^1(\mu))^T\}\\
&\quad + \mathrm{tr}\{D_i^1(\mu)\varepsilon_i\Omega_{i,\mathrm{pri}}^1(\mu)\varepsilon_i^T(D_i^1(\mu))^T\}\\
&\quad + \mathrm{tr}\{K_i^1(\mu)\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu)\Omega_{i,\mathrm{pri}}^1(\mu)(\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu))^T(K_i^1(\mu))^T\}\\
&\quad + 2\,\mathrm{tr}\{K_i^1(\mu)\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu)\Omega_{i,\mathrm{pri}}^1(\mu)\varepsilon_i^T(D_i^1(\mu))^T\}\\
&\quad + 2\,\mathrm{tr}\{G_i^1(\mu)(D_i^1(\mu))^T\} - 2\,\mathrm{tr}\{K_i^1(\mu)\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu)G_i^1(\mu)(D_i^1(\mu))^T\}\\
&\quad - 2\,\mathrm{tr}\{D_i^1(\mu)\varepsilon_iG_i^1(\mu)(D_i^1(\mu))^T\} + \mathrm{tr}\{D_i^1(\mu)U_i^1(\mu)(D_i^1(\mu))^T\}\\
&\quad + \mathrm{tr}\{K_i^1(\mu)[(I - \bar{\varpi}_1(\mu))\bar{\varpi}_1(\mu)\circ(\bar{C}_i^1(\mu)X(\mu)(\bar{C}_i^1(\mu))^T) + \bar{\varpi}_1(\mu)\circ\bar{R}_i^1(\mu)](K_i^1(\mu))^T\}
\end{aligned}
\tag{A40}
$$

$$
\begin{aligned}
0 = &-2\Omega_{i,\mathrm{pri}}^1(\mu)(\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu))^T + 2K_i^1(\mu)\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu)\Omega_{i,\mathrm{pri}}^1(\mu)(\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu))^T\\
&+ 2D_i^1(\mu)\varepsilon_i\Omega_{i,\mathrm{pri}}^1(\mu)(\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu))^T\\
&+ 2K_i^1(\mu)[(I - \bar{\varpi}_1(\mu))\bar{\varpi}_1(\mu)\circ(\bar{C}_i^1(\mu)X(\mu)(\bar{C}_i^1(\mu))^T) + \bar{\varpi}_1(\mu)\circ\bar{R}_i^1(\mu)]\\
&- 2D_i^1(\mu)(G_i^1(\mu))^T(\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu))^T
\end{aligned}
\tag{A41}
$$

$$
\begin{aligned}
K_i^1(\mu)\Psi_i^1(\mu) &= \Omega_{i,\mathrm{pri}}^1(\mu)(\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu))^T - D_i^1(\mu)\varepsilon_i\Omega_{i,\mathrm{pri}}^1(\mu)(\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu))^T\\
&\quad + D_i^1(\mu)(G_i^1(\mu))^T(\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu))^T
\end{aligned}
\tag{A42}
$$

$$
\begin{aligned}
0 = &-2\Omega_{i,\mathrm{pri}}^1(\mu)\varepsilon_i^T + 2K_i^1(\mu)\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu)\Omega_{i,\mathrm{pri}}^1(\mu)\varepsilon_i^T + 2D_i^1(\mu)\varepsilon_i\Omega_{i,\mathrm{pri}}^1(\mu)\varepsilon_i^T\\
&+ 2D_i^1(\mu)U_i^1(\mu) + 2G_i^1(\mu) - 2K_i^1(\mu)\bar{\varpi}_1(\mu)\bar{C}_i^1(\mu)G_i^1(\mu)\\
&- 2D_i^1(\mu)\varepsilon_iG_i^1(\mu) - 2D_i^1(\mu)(G_i^1(\mu))^T\varepsilon_i^T
\end{aligned}
\tag{A43}
$$

$$
\begin{aligned}
A(z) &= \begin{bmatrix} 2.17 & 1.48 \\ -1.76 & -0.85 \end{bmatrix}, \quad B(z) = \begin{bmatrix} 0.24 \\ 0.51 \end{bmatrix},\\
C_i(z) &= \begin{bmatrix} 3i & 15 \\ 20 & 10 \end{bmatrix}, \quad D_i(z) = \begin{bmatrix} 0.02 \\ 0.1 \end{bmatrix}, \quad i = 1, 3, 5,\\
C_i(z) &= \begin{bmatrix} 2i & 10 \\ 15 & 12 \end{bmatrix}, \quad D_i(z) = \begin{bmatrix} 0.2 \\ 0.01 \end{bmatrix}, \quad i = 2, 4, 6
\end{aligned}
\tag{A44}
$$