Numerical Method for Electromagnetic Wave Propagation Problem in a Cylindrical Anisotropic Waveguide with Longitudinal Magnetization

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Abstract

The propagation of monochromatic electromagnetic waves in metal circular cylindrical dielectric waveguide with longitudinal magnetization filled with anisotropic inhomogeneous waveguide is considered. The physical problem is reduced to solving a transmission eigenvalue problem for a system of ordinary differential equations. Spectral parameters of the problem are propagation constants of the waveguide. Numerical results are obtained using a modification of the projecting methods. The comparison with known exact solutions (for particular values of parameters) are made.

1. Introduction

A large class of vector electromagnetic problem concerns electromagnetic wave propagation. The constitutive parameters ε and μ of standard dielectric and magnetic media are determined by their physical structure. However, the media with unusual properties are often required which can be obtained using either dielectrics that are uniform or partially filled. The parameters of such media depend on the mutual position of the particles and may be anisotropic [1]. It is known also that the permittivity of a dielectric (or the permeability of a magnetic) may depend on the radial coordinate [2]. The primary goal here is to construct a numerical method to determine the spectrum of normal electromagnetic waves that propagate in such structures.

Numerical methods for calculating the parameters of various types of waveguide structures are described in the monographs and review papers [3, 4, 5, 6]. However, it should be said that most of the methods applied to homogeneous waveguides, are not common and are difficult to implement and apply for specific inhomogeneous and/or anisotropic structures.

In this work the wave propagation in inhomogeneous metal-dielectric anisotropic cylindrical waveguides is studied numerically using the modification of the projection methods [7].

2. Statement of the problem

Consider three-dimensional space $\mathbb{R}^3$ with a cylindrical coordinate system $O\rho\varphi z$ filled with isotropic medium having constant permittivity $\varepsilon = \varepsilon_0$ ($\varepsilon_0 > 0$ is the permittivity of free space), and constant permeability $\mu = \mu_0$ (where $\mu_0 > 0$ is the permeability of free space).

A metal dielectric circular cylindrical waveguide Σ filled with anisotropic inhomogeneous medium is placed parallel to the axis Oz. The waveguide Σ has a cross section

$$\Sigma := \{(\rho, \varphi, z) : r_0 \le \rho \le r, 0 \leqslant \varphi < 2\pi\}$$

and its generating line (the waveguide axis) is parallel to the axis Oz (see. Fig. 1).

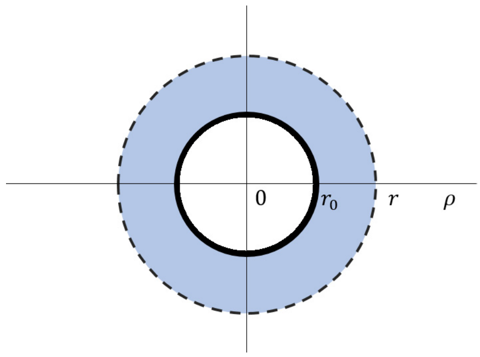

Figure 1: The cylindrical anisotropic waveguide Σ.

We will consider monochromatic waves

$$\mathbf{E}e^{-i\omega t} = e^{-i\omega t}\left(E_\rho, E_\varphi, E_z\right)^T,$$
$$\mathbf{H}e^{-i\omega t} = e^{-i\omega t}\left(H_\rho, H_\varphi, H_z\right)^T,$$

where $(\,\cdot\,)^T$ denotes the transpose operation. Each component of the field $\mathbf{E}, \mathbf{H}$ is a function of three spatial variables.

Complex amplitudes of the electromagnetic field $\mathbf{E}, \mathbf{H}$ satisfy the Maxwell equations

$$\begin{cases} \mathrm{rot}\mathbf{H} = -\mathrm{i}\omega\varepsilon\mathbf{E}, \\ \mathrm{rot}\mathbf{E} = \mathrm{i}\omega\mu\mathbf{H}, \end{cases} \tag{1}$$

subject to the following boundary conditions. The tangential components of the electric field vanish on the metal surface $\rho = r_0$; tangential field components are continuous on the media interface $\rho = r$; the complex amplitudes obey the radiation condition at infinity: the electromagnetic field decays as $O(|\rho|^{-1})$ when $\rho \to \infty$. The permittivity ε inside the waveguide is constant; the permeability $\hat{\mu}$ is specified by the expression

$$\hat{\mu} = \begin{bmatrix} \mu_\rho & -i\mu_\varphi & 0 \\ i\mu_\varphi & \mu_\rho & 0 \\ 0 & 0 & \mu_z \end{bmatrix}, \tag{2}$$

where $\mu_\rho(\rho)$, $\mu_\varphi(\rho)$ and $\mu_z(\rho)$ are sufficiently smooth functions which depend on the radial coordinate ρ.

The surface waves propagating along the axis Oz of the waveguide Σ have the form [8]

$$\begin{aligned} E_\rho &= \mathrm{E}_\rho(\rho)e^{i\gamma z}, & H_\rho &= \mathrm{H}_\rho(\rho)e^{i\gamma z} \\ E_\varphi &= \mathrm{E}_\varphi(\rho)e^{i\gamma z}, & H_\varphi &= \mathrm{H}_\varphi(\rho)e^{i\gamma z}, \\ E_z &= \mathrm{E}_z(\rho)e^{i\gamma z}, & H_z &= \mathrm{H}_z(\rho)e^{i\gamma z}, \end{aligned} \tag{3}$$

where γ is the real propagation constant (spectral parameter of the problem). In what follows we often omit the arguments of functions when it does not lead to misunderstanding.

3. Differential equations

Inside the waveguide $\mu = \hat{\mu}$ and $\varepsilon = \varepsilon$. Substituting $\mathbf{E}$ and $\mathbf{H}$ with components (3) into equations (1), we obtain

$$\begin{cases} i\gamma H_\varphi = i\omega\varepsilon E_\rho, \\ i\gamma H_\rho - H_z' = -i\omega\varepsilon E_\varphi, \\ \dfrac{1}{\rho}(\rho H_\varphi)' = -i\omega\varepsilon E_z, \\ i\gamma E_\varphi = -i\omega\mu_\rho H_\rho - \omega\mu_\varphi H_\varphi, \\ i\gamma E_\rho - E_z' = -\omega\mu_\varphi H_\rho + i\omega\mu_\rho H_\varphi, \\ \dfrac{1}{\rho}(\rho E_\varphi)' = i\omega\mu_z H_z, \end{cases} \tag{4}$$

where the prime denotes differentiation w.r.a ρ. Expressing the functions E_ρ, E_z, H_ρ and H_z through E_φ and H_φ from the 1st, 3rd, 4th and 6th equation of system (4), we find

$$\begin{aligned} H_\rho &= \frac{-\gamma E_\varphi + \omega\mu_\varphi i H_\varphi}{\omega\mu_\rho}, & E_\rho &= \frac{\gamma H_\varphi}{\omega\varepsilon}, \\ H_z &= -\frac{(\rho i E_\varphi)'}{\omega\mu_z\rho}, & E_z &= \frac{(\rho i H_\varphi)'}{\omega\varepsilon\rho}. \end{aligned} \tag{5}$$

Substituting the expressions for E_ρ, E_z, H_ρ and H_z into the 2nd and 5th equations of system (5) and introducing the notation $u_e := i\rho E_\varphi(\rho)$, $u_m := \rho H_\varphi(\rho)$, we obtain

$$\begin{aligned} L_e u_e &= u_e'' - p_e u_e' + \left(q_e - h_e\gamma^2\right) u_e = \gamma f_e u_m, \\ L_m u_m &= u_m'' - p_m u_m' + \left(q_m - h_m\gamma^2\right) u_m = \gamma f_m u_e \end{aligned} \tag{6}$$

where

$$p_e = \frac{(\rho\mu_z)'}{\rho\mu_z}, \qquad p_m = \frac{1}{\rho},$$

$$q_e = \omega^2\varepsilon\mu_z, \qquad q_m = \omega^2\varepsilon\frac{\mu_\rho^2 - \mu_\varphi^2}{\mu_\rho},$$

$$h_e = \frac{\mu_z}{\mu_\rho}, \qquad h_m = 1,$$

$$f_e = \omega\frac{\mu_\varphi\mu_z}{\mu_\rho}, \qquad f_m = \omega\frac{\varepsilon\mu_\varphi}{\mu_\rho}.$$

Outside the waveguide, where $\mu = \mu_0 = 1$ and $\varepsilon = \varepsilon_0 = 1$, system (1) takes the form of Bessel's equations

$$\left(\frac{u_e'}{\rho}\right)' - \frac{k_1^2 u_e}{\rho} = 0,$$
$$\left(\frac{u_m'}{\rho}\right)' - \frac{k_1^2 u_m}{\rho} = 0,$$

with general solutions

$$\begin{aligned} u_e &= C_1\rho I_1(k_1\rho) + C_2\rho K_1(k_1\rho), \ \rho > r \\ u_m &= C_3\rho I_1(k_1\rho) + C_4\rho K_1(k_1\rho), \ \rho > r \end{aligned} \tag{7}$$

where $k_1^2 = \gamma^2 - \omega^2$, I_1 is the modified Bessel function and K_1 is the Macdonald function [9] and $C_1, ..., C_4$ denote arbitrary constants.

The solutions (7) takes a form

$$\begin{aligned} u_e &= C_1\rho K_1(k_1\rho), \ \rho > r \\ u_m &= C_2\rho K_1(k_1\rho), \ \rho > r \end{aligned} \tag{8}$$

where the radiation condition at infinity is taken into account.

4. Transmission conditions and transmission problem

Tangential components of the electromagnetic field are known to be continuous at the interface. In this case the tangential components are E_φ, E_z, H_φ and H_z. Thus we obtain the following transmission conditions for u_e and u_m

$$\begin{gathered} u_e(r_0) = 0, \, u_m'(r_0) = 0, \\ u_e(r-0) = u_e(r+0), \, u_m(r-0) = u_m(r+0), \\ \frac{u_e'(r-0)}{\mu_z(r-0)} = \frac{u_e'(r+0)}{\mu_0}, \, \frac{u_m'(r-0)}{\varepsilon} = \frac{u_m'(r+0)}{\varepsilon_0}, \end{gathered} \tag{9}$$

where $[v]|_{\rho=s} = \lim\limits_{\rho\to s-0} v(\rho) - \lim\limits_{\rho\to s+0} v(\rho)$ is the jump of the limit values of the function at the point s.

The main problem considered in this study is formulated as follows: ***Problem*** P*: to find* $\hat{\gamma}$ *such that there exist non-trivial functions* $u_e(\rho;\hat{\gamma})$ *and* $u_m(\rho;\hat{\gamma})$ *satisfying system* (6)*, transmission conditions* (9)*, and having the form* (8) *outside the waveguide.*

5. Variation formulation

Let us give the variational formulation of the problem P. Using the first Green's formula, we obtain

$$\int_{r_0}^{r} vLud\rho =$$

$$= \int_{r_0}^{r} vu''d\rho - \int_{r_0}^{r} vpu'd\rho + \int_{r_0}^{r} v\left(q - h\gamma^2\right)ud\rho =$$

$$= u'v|_{r_0}^{r} - \int_{r_0}^{r} u'v'd\rho - \int_{r_0}^{r} pu'vd\rho + \int_{r_0}^{r}\left(q - h\gamma^2\right)uvd\rho =$$

$$= -\gamma^2\int_{r_0}^{r} huvd\rho - \int_{r_0}^{r} u'v'd\rho -$$

$$- \int_{r_0}^{r} pu'vd\rho + \int_{r_0}^{r} quvd\rho + u'(r)v(r), \quad (10)$$

where $u = u_j,\ h = h_j, p = p_j, q = q_j,\ j = e$ or m.

Let us consider the smooth test functions v_e and v_m.

Note 1. *We assume that the test functions v_e and v_m satisfy the following conditions*

$$v_e(r_0) = 0,\ \ v_e(r) = 1,$$
$$v_m'(r_0) = 0,\ \ v_m(r) = 1,$$

which coincide with conditions for functions u_e and u_m at the boundary r_0.

Multiplying the left and right sides of equations (10) by the test functions v_e and v_m, respectively, and summing up we obtain

$$\int_{r_0}^{r}(v_eL_eu_e + v_mL_mu_m)d\rho =$$

$$= -\gamma^2\int_{r_0}^{r}(h_eu_ev_e + h_mu_mv_m)d\rho -$$

$$- \int_{r_0}^{r}(u_e'v_e' + u_m'v_m')d\rho - \int_{r_0}^{r}(p_eu_e'v_e + u_m'v_m)d\rho +$$

$$+ \int_{r_0}^{r}(q_eu_ev_e + q_mu_mv_m)d\rho +$$

$$+ u_e'(r)v_e(r) + u_m'(r)v_m(r). \quad (11)$$

Taking into account the right-hand sides of the equations of system (6), we have

$$\int_{r_0}^{r}(v_eL_eu_e + v_mL_mu_m)d\rho =$$

$$= \gamma\int_{r_0}^{r}(f_eu_mv_e + f_mu_ev_m)d\rho. \quad (12)$$

From (9) we determine u_e' and u_m' at the point r

$$u_e'(r) = -k_1\frac{\mu_z(r)}{\mu_0}\frac{K_0(k_1r)}{K_1(k_1r)}u_e(r), \quad (13)$$

$$u_m'(r) = -k_1\frac{\varepsilon}{\varepsilon_0}\frac{K_0(k_1r)}{K_1(k_1r)}u_m(r). \quad (14)$$

From (11) taking into account (12) and (13), we obtain the *variational equation*

$$\gamma^2\int_{r_0}^{r}(h_eu_ev_e + h_mu_mv_m)d\rho +$$

$$+ \int_{r_0}^{r}(u_e'v_e' + u_m'v_m')d\rho + \int_{r_0}^{r}(p_eu_e'v_e + p_mu_m'v_m)d\rho -$$

$$- \int_{r_0}^{r}(q_eu_ev_e + q_mu_mv_m)d\rho +$$

$$+ k_1\frac{K_0(k_1r)}{K_1(k_1r)}\left(\frac{\mu_z(r)}{\mu_0}u_e(r)v_e(r) + \frac{\varepsilon}{\varepsilon_0}u_m(r)v_m(r)\right) +$$

$$+ \gamma\int_{r_0}^{r}(f_eu_mv_e + f_mu_ev_m)d\rho,\ \forall v_e,\ v_m, \quad (15)$$

which hold for any test functions v_e and v_m. The solution of (15) is equivalent to the original problem P.

6. Projection method

Using the projection method [10] let us reduce the variational equation (15) to a system of algebraic equations. Firstly, split an interval $[r_0,\ r]$ into n subintervals with the length

$$l = \frac{r_0 - r}{n}.$$

Let us define a set of n subintervals

$$\Phi_i = [r_0 + (i-1)l,\ r_0 + (i+1)l],\ i = 1, ..., n-1$$

and

$$\Phi_n = [r_0 + (n-1)l,\ h],$$

and set of $n+1$ subintervals

$$\Psi_1 = [r_0,\ r_0 + l],$$

$$\Psi_j = [r_0 + (i-2)l,\ r_0 + il],\ j = 2, ..., n$$

and

$$\Psi_{n+1} = [r_0 + (n-1)l,\ h].$$

These subintervals we call *base finite elements*.

In accordance with the scheme of the projection method, it is necessary to introduce *basis functions* ϕ_i and ψ_j in order to approximate the solution. The basis functions are defined on each subinterval Φ_i and Ψ_j (ϕ_i and ψ_j vanishes outside the intervals Φ_i and Ψ_j, respectively).

The basis functions ϕ_i define on Φ_i, are

$$\phi_i = \begin{cases} \dfrac{\rho - r_0 - (i-1)l}{l}, & \rho < r_0 + il, \\ -\dfrac{\rho - r_0 - (i+1)l}{l}, & \rho > r_0 + il, \end{cases}, i = \overline{1, n-1}$$

and

$$\phi_n = \frac{\rho - h + l}{l};$$

The basis functions ψ_i defined on Φ_i are

$$\psi_1 = -\frac{\rho^2 - 2r_0\rho + r_0^2 - l^2}{l^2},$$

$$\psi_2 = \begin{cases} \dfrac{\rho^2 - 2r_0\rho + r_0^2}{l^2}, & \rho < r_0 + l, \\ -\dfrac{\rho - r_0 - 2l}{l}, & \rho > r_0 + l, \end{cases}$$

$$\psi_j = \begin{cases} \dfrac{\rho - r_0 - (i-2)l}{l}, & \rho < r_0 + (i-1)l, \\ -\dfrac{\rho - r_0 - il}{l}, & \rho > r_0 + (i-1)l, \end{cases}, j = \overline{3, n}$$

and

$$\psi_{n+1} = \frac{\rho - h + l}{l}.$$

Such defined basis functions takes into account the physical nature of the problem under consideration.

We assume an approximate solution with real coefficients α_i and β_j such that

$$u_e = \sum_{i=1}^{n} \alpha_i \phi_i, \quad u_m = \sum_{j=1}^{n+1} \beta_j \psi_j. \tag{16}$$

Substituting functions u_e and u_m with representations (16) into the variational equation (15), we obtain a system of linear equations with respect to α_i and β_j (for fixed value of γ)

$$A(\gamma)x = 0, \tag{17}$$

where matrices $A(\gamma)$ and x have the form

$$A = \begin{pmatrix} A_{ee}^{1,1} & \cdots & A_{ee}^{1,n} & A_{em}^{1,1} & \cdots & A_{em}^{1,n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{ee}^{n,1} & \cdots & A_{ee}^{n,n} & A_{em}^{n,1} & \cdots & A_{em}^{n,n+1} \\ A_{me}^{1,1} & \cdots & A_{me}^{1,n} & A_{mm}^{1,1} & \cdots & A_{mm}^{1,n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{me}^{n+1,1} & \cdots & A_{me}^{n+1,n} & A_{mm}^{n+1,1} & \cdots & A_{mm}^{n+1,n+1} \end{pmatrix},$$

$$x = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \\ \beta_1 \\ \vdots \\ \beta_{n+1} \end{pmatrix},$$

where

$$A_{ee}^{i,j} = \gamma^2 \int\limits_{\Phi_i} h_e \phi_i \phi_j d\rho + \int\limits_{\Phi_i} \phi_i' \phi_j' d\rho + \int\limits_{\Phi_i} p_e \phi_i' \phi_j d\rho - \int\limits_{\Phi_i} q_e \phi_i \phi_j d\rho + k_1 \frac{\mu_z(r)}{\mu_0} \frac{K_0(k_1 r)}{K_1(k_1 r)} \phi_i(r)\phi_j(r), \ i,j = \overline{1,n};$$

$$A_{em}^{i,j} = \gamma \int\limits_{\Phi_i} f_m \phi_i \psi_j d\rho, \ i = \overline{1,n}, j = \overline{1,n+1},$$

$$A_{me}^{i,j} = \gamma \int\limits_{\Psi_i} f_e \psi_i \phi_j d\rho, \ i = \overline{1,n+1}, j = \overline{1,n},$$

$$A_{mm}^{i,j} = \gamma^2 \int\limits_{\Psi_i} h_m \psi_i \psi_j d\rho + \int\limits_{\Psi_i} \psi_i' \psi_j' d\rho + \int\limits_{\Psi_i} p_m \psi_i' \psi_j d\rho - \int\limits_{\Psi_i} q_m \psi_i \psi_j d\rho + k_1 \frac{\varepsilon}{\varepsilon_0} \frac{K_0(k_1 r)}{K_1(k_1 r)} \psi_i(r)\psi_j(r), \ i,j = \overline{1,n+1}.$$

Thus $A(\gamma)$ is a $(2n+1)\times(2n+1)$ matrix. Let us denote by $\Delta(\gamma)$ the determinant of $A(\gamma)$

$$\Delta(\gamma) = \det A(\gamma). \tag{18}$$

Note 2. *If there exists $\gamma = \widetilde{\gamma}$ such that $\Delta(\widetilde{\gamma}) = 0$, then $\widetilde{\gamma}$ is an approximate spectral parameter of Problem P. In other words, if an interval $[\underline{\gamma}, \ \gamma]$ is such that $\Delta(\underline{\gamma}) \times \Delta(\gamma) < 0$, then this means that there exists $\gamma = \widetilde{\gamma} \in [\underline{\gamma}, \ \gamma]$ which is an spectral parameter of Problem P. This value can be calculated with any prescribed accuracy.*

7. Numerical results

The results of the numerical solution of the problem of propagating electromagnetic waves of an anisotropic magnetic waveguide structure are presented. Numerical results are obtained with the help of the shooting method. Radii of the waveguide (internal and external) $r_0 = 2$ cm, $r = 4$ cm, permittivity $\varepsilon = 4$. The values of the tensor components $\hat{\mu}$, are shown in the figure captions.

Numerical analysis of the behavior of dispersion curves (graphs of the dependence of the propagation constant γ on the circular frequency ω) is performed for the different components of tensor $\hat{\mu}$. In the case of $\mu_\varphi \to 0$, the number of hybrid modes coincides with the sum of the "polarized" modes (TE and TM), the dispersion curves for $\mu_\varphi = 0$ (Fig. 2) coincide with the known dispersion curves for problems on propagating TE–and TM–polarized waves of an metal-dielectric waveguide [11].

Figure 3 and Figure 4 shows the dispersion curves for the case when components of tensor $\hat{\mu}$ are functions.

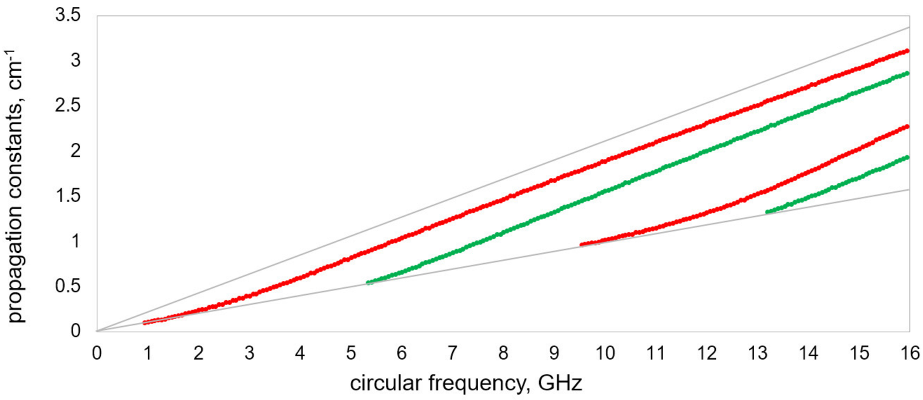

Figure 2: Dispersion curves for the values of the components of the tensor $\hat{\mu}$: $\mu_\rho = 1,\ \mu_z = 1;\ \mu_\varphi = 0$. The red curves correspond to TM-polarized waves, the green curves to TE-polarized waves.

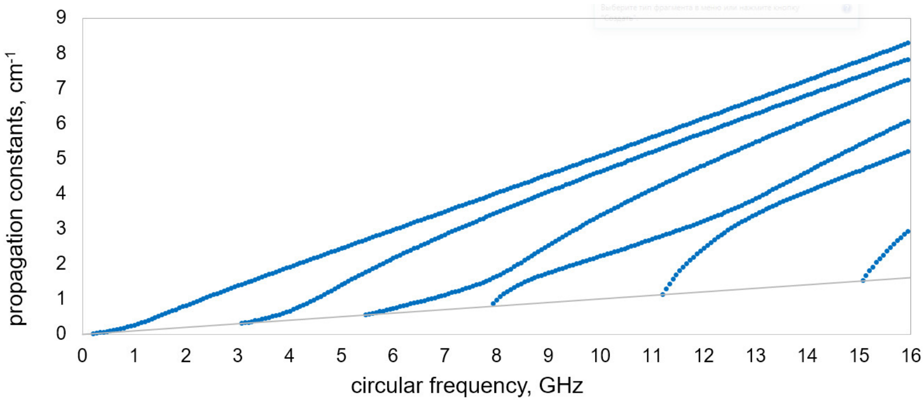

Figure 3: Dispersion curves for the values of the components of the tensor $\hat{\mu}$: $\mu_\rho = 1 + \rho,\ \mu_z = 1;\ \mu_\varphi = 1$. The blue curves correspond to hybrid waves.

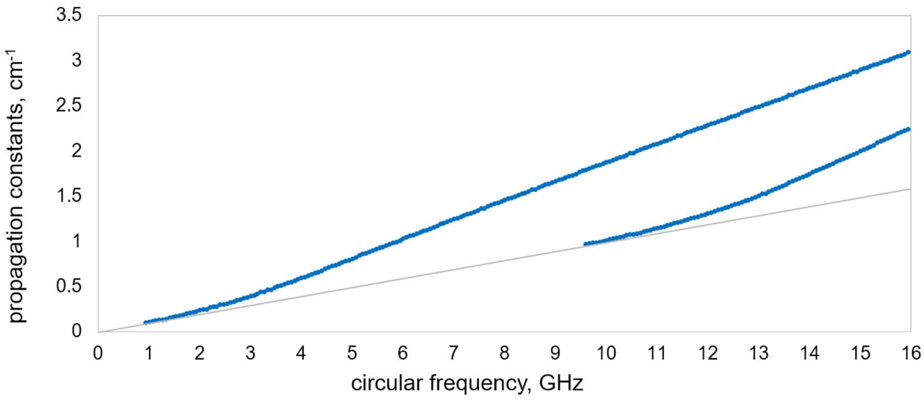

Figure 4: Dispersion curves for the values of the components of the tensor $\hat{\mu}$: $\mu_\rho = 1,\ \mu_z = 1 + \rho;\ \mu_\varphi = 0.125$. The blue curves correspond to hybrid waves.

8. Conclusion

This work continues the investigation of the spectrum of metal dielectric waveguides with inhomogeneous filling. The paper[7] presents a numerical method for solving the problem of propagating waves of a dielectric waveguide. This method was used to numerically study the spectrum of a waveguide filled with an inhomogeneous anisotropic magnetic medium (ferrite). The method allows us to determine approximate eigenvalues with any prescribed accuracy. The approach described in this paper can be applied to other problems, e.g., to multilayered opened waveguides.

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