Detailed Analysis of the Electromagnetic Normal Modes of Spherical and Annular Spherical Cavities: Energy, Thrust and Losses. Implications on the Spherical Casimir Effect

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ABSTRACT This paper develops analytical expressions of energy, thrust and losses for all electromagnetics normal modes in spherical and annular spherical cavities. The implications on the spherical Casimir effect are also investigated.

INDEX TERMS Annular, Casimir effect, Cavity, Electromagnetic, Normal modes, Spherical, Thrust.

I. INTRODUCTION

THE energy and losses of spherical cavities were studied in [1] for two of the modes and in [2] for six of them. In this paper, we extend the analytical calculation to all modes, both for spherical cavities and for annular spherical cavities. We add the calculation of the thrust exerted on the walls of the cavities, with the aim of being able to use the present study for a more direct calculation of the spherical Casimir effect than the classical calculation [3].

A. PRELIMINARY WORK

The present study required the development of new formulas for integrals containing squares of special functions. These formulas were established in a preliminary work [4] and given here in appendix.

B. CHOICE OF THE EXPRESSIONS OF FIELDS

The explicit expression of the spherical electromagnetic wave fields can be obtained by following the detailed method provided in [5]. The result is given in [2] in the particular case where the domain considered includes the origin. To obtain a general formulation, it is sufficient to replace in the expressions provided in [2] the first kind Bessel function by a combination of these functions with the second kind Bessel function [4]. We will write this combination here in the form:

$$\Psi_\nu(x) = J_\nu(x)\cos(\zeta) - Y_\nu(x)\sin(\zeta) \tag{1}$$

where ζ is any parameter in the range $[0, \pi[$. In the particular case of a spherical cavity, the second term of (1) must be zero to avoid having a singularity of the fields at the center of the cavity. In this case, we must therefore have

$$\zeta = 0 \; and \; thus \; \Psi_\nu(x) = J_\nu(x) \tag{2}$$

The writing (1) is interesting because, for large values of x, using the asymptotic expansions [6](9.2.1 and 9.2.2) of $J_\nu(x)$ and $Y_\nu(x)$, we obtain the asymptotic expansion

$$\Psi_\nu(x) \approx \sqrt{\frac{2}{\pi x}}\cos(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi + \zeta) \; if \; x \; is \; large \tag{3}$$

In this paper, we use Bessel functions of fractional order $\nu = \ell + 1/2$ with $\ell \in \mathbb{Z}$. These functions multiplied by $\sqrt{2\pi/x}$ are most commonly known as spherical Bessel functions.

C. NOTATIONS

We use the spherical coordinates r , θ and φ which form a direct system for this order. The components of the vectors are given in the orthonormal reference frame associated with these coordinates. The angle dependence of the waves with respect to θ and φ is fixed by two integers which take respectively the values:

$$\ell = 1, 2, ..., \infty \tag{4}$$

and

$$m = 0, 1, ..., \ell \tag{5}$$

The dependency on the coordinate φ is set by the number m in the form of $\cos(m\varphi)$ and $\sin(m\varphi)$. When $m > 1$, we can distinguish two very similar waves obtained by permuting these two functions, i.e., an even wave and an odd wave. In what follows, we will consider only one of these waves. A distinction is made between TE waves (transverse electric) and TM waves (transverse magnetic).

We study the normal modes of annular spherical cavities, i.e., between two concentric conducting spheres of radius a and R:

$$0 < a < r < R < \infty \tag{6}$$

The case of spherical cavities will be obtained in the limit $a \to 0$. In a cavity, in addition to the numbers ℓ and m, one introduces a number

$$n = 1, 2, ...\infty \quad (7)$$

to number the normal TE or TM modes in order of increasing frequency. The modes are therefore all completely defined, apart from parity, by the designation $TE_{\ell mn}$ or $TM_{\ell mn}$. In the following, the time dependence in not made explicit. To specify a 90° phase, we use the complex number

$$j = \sqrt{-1} \quad (8)$$

The spatial dependence will be written using the wave number k which is related to the pulsation ω by the relation

$$k = \frac{\omega}{c} \quad (9)$$

where c is the speed of light in the considered medium

$$c = \frac{1}{\sqrt{\epsilon\mu}} \quad (10)$$

so

$$k^2 = \omega^2\epsilon\mu \quad (11)$$

To simplify the writing, we also use the coefficient

$$M(m) = 1\ if\ m \neq 0\ and\ M(m) = 2\ if\ m = 0 \quad (12)$$

commonly known as Neumann factor with another notation. With this coefficient, we can write without worrying about the value of m

$$\int_0^{2\pi} \cos^2(m\varphi) = M\pi \quad (13)$$

and

$$\int_0^{2\pi} m\sin^2(m\varphi) = mM\pi \quad (14)$$

II. STUDY OF MODES TE

A. EXPRESSION OF FIELDS

Apart from parity, the fields of waves $TE_{\ell mn}$ are of the form given in [2], i.e., after generalization:

$$E_r = 0 \quad (15)$$

$$E_\theta = \frac{mA}{\sqrt{r}\sin\theta}\Psi_{\ell+1/2}(kr)P_\ell^m(\cos(\theta))\sin(m\varphi) \quad (16)$$

$$E_\varphi = \frac{A}{\sqrt{r}}\Psi_{\ell+1/2}(kr)\frac{d}{d\theta}[P_\ell^m(\cos\theta)]\cos(m\varphi) \quad (17)$$

and

$$H_r = \frac{\ell(\ell+1)A}{j\omega\mu r^{3/2}}\Psi_{\ell+1/2}(kr)P_\ell^m(\cos\theta)]\cos(m\varphi) \quad (18)$$

$$H_\theta = \frac{A}{j\omega\mu r}\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\frac{d}{d\theta}[P_\ell^m(\cos\theta)]\cos(m\varphi) \quad (19)$$

$$H_\varphi = \frac{mA}{j\omega\mu r\sin\theta}\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]P_\ell^m(\cos\theta)\sin(m\varphi) \quad (20)$$

B. BOUNDARY CONDITIONS

Since the walls of the cavity are perfectly conductive, in $r = a$ and $r = R$, the tangential component of the electric field and the radial component of the magnetic field must be zero. It is immediately clear that these conditions are satisfied if

$$\Psi_{\ell+1/2}(ka) = 0 \quad (21)$$

and

$$\Psi_{\ell+1/2}(kR) = 0 \quad (22)$$

Considering (1), equations (21) and (22) can be written:

$$\tan(\zeta) = \frac{J_{\ell+1/2}(ka)}{Y_{\ell+1/2}(ka)} \quad (23)$$

and

$$\tan(\zeta) = \frac{J_{\ell+1/2}(kR)}{Y_{\ell+1/2}(kR)} \quad (24)$$

Identifying (23) and (24) gives an equation defining the values of k. That equation is equivalent with the similar condition given in [3]. The two conditions (21) (22) are thus satisfied for a discrete set of pairs (k, ζ). From (23) and (24), we also obtain, with $\pm 1 = sgn[J_{\ell+1/2}(ka)]$:

$$\cos(\zeta) = \frac{\pm Y_{\ell+1/2}(ka)}{\sqrt{J_{\ell+1/2}^2(ka) + Y_{\ell+1/2}^2(ka)}} \quad (25)$$

$$\sin(\zeta) = \frac{\pm J_{\ell+1/2}(ka)}{\sqrt{J_{\ell+1/2}^2(ka) + Y_{\ell+1/2}^2(ka)}} \quad (26)$$

and, with $\pm 1 = sgn[J_{\ell+1/2}(kR)]$:

$$\cos(\zeta) = \frac{\pm Y_{\ell+1/2}(kR)}{\sqrt{J_{\ell+1/2}^2(kR) + Y_{\ell+1/2}^2(kR)}} \quad (27)$$

$$\sin(\zeta) = \frac{\pm J_{\ell+1/2}(kR)}{\sqrt{J_{\ell+1/2}^2(kR) + Y_{\ell+1/2}^2(kR)}} \quad (28)$$

We can obtain an interesting expression of the derivative $\Psi'_{\ell+1/2}(ka)$. Using (1), we have

$$\Psi'_{\ell+1/2}(ka) = J'_{\ell+1/2}(ka)\cos\zeta - Y'_{\ell+1/2}(ka)\sin\zeta \quad (29)$$

Introducing (25) and (26) in (29), it comes

$$\Psi'_{\ell+1/2}(ka) = \mp\frac{J_{\ell+1/2}(ka)Y'_{\ell+1/2}(ka) - J'_{\ell+1/2}(ka)Y_{\ell+1/2}(ka)}{\sqrt{J_{\ell+1/2}^2(ka) + Y_{\ell+1/2}^2(ka)}} \quad (30)$$

The numerator of (30) is a Wronskian. Using its expression [6](9.1.16)(9.1.27), we obtain

$$\Psi'_{\ell+1/2}(ka) = \mp\frac{2}{\pi ka}\frac{1}{\sqrt{J_{\ell+1/2}^2(ka) + Y_{\ell+1/2}^2(ka)}} \quad (31)$$

Of course, one has the similar expression for $r = R$:

$$\Psi'_{\ell+1/2}(kR) = \mp\frac{2}{\pi kR}\frac{1}{\sqrt{J_{\ell+1/2}^2(kR) + Y_{\ell+1/2}^2(kR)}} \quad (32)$$

C. CALCULATION OF ENERGY VIA THE ELECTRIC FIELD

The simplest way to calculate the energy of a TE normal mode is to do the calculation at the moment when the magnetic field is zero, since at that moment all energy is in electric form and the electric field (15) (16) (17) has only two non-zero components, namely (16) and (17). By introducing (15), (16) and (17) into the expression for electric energy

$$W = \int\int\int \frac{\epsilon}{2} E^2 r^2 \sin\theta dr d\theta d\varphi \tag{33}$$

and performing the integrals with respect to φ using (13) and (14), we obtain

$$W = \frac{\epsilon\pi A^2}{2} M \int\int \{ \frac{m^2}{r\sin^2\theta} \Psi^2_{\ell+1/2}(kr)[P^m_\ell(\cos\theta)]^2 + \frac{1}{r}\Psi^2_{\ell+1/2}(kr)[\frac{d}{d\theta}P^m_\ell(\cos\theta)]^2\} r^2 \sin\theta dr d\theta \tag{34}$$

or

$$W = \frac{\epsilon\pi A^2}{2} M[\int_a^R r\Psi^2_{\ell+1/2}(kr)dr] \int_0^\pi \{\frac{m^2}{\sin\theta}[P^m_\ell(\cos\theta)]^2 + \sin\theta[\frac{d}{dr}P^m_\ell(\cos\theta)]^2\}d\theta \tag{35}$$

The integral with respect to θ is obtained using (168) and (167).

$$W = \frac{\epsilon\pi A^2}{2} M[\int_a^R r\Psi^2_{\ell+1/2}(kr)dr] (m + \frac{\ell(\ell+1)}{\ell+1/2} - m)\frac{(\ell+m)!}{(\ell-m)!} \tag{36}$$

or

$$W = \frac{\epsilon\pi A^2}{2} M[\int_a^R r\Psi^2_{\ell+1/2}(kr)dr] \frac{\ell(\ell+1)}{(\ell+1/2)} \frac{(\ell+m)!}{(\ell-m)!} \tag{37}$$

The integral with respect to r is a Lommel integral. It is done by using (165), then simplifying the result given the boundary conditions (21) (22). We obtain

$$W = \frac{\epsilon\pi A^2}{2} M[R^2\Psi'^2_{\ell+1/2}(kR) - a^2\Psi'^2_{\ell+1/2}(ka)] \frac{\ell(\ell+1)}{\ell+1/2}\frac{(\ell+m)!}{(\ell-m)!} \tag{38}$$

D. CALCULATION OF ENERGY VIA THE MAGNETIC FIELD

The energy expression can also be obtained by integrating the magnetic energy taken at the moment when the electric field cancels. The calculation is more difficult than in the previous paragraph because the magnetic field has three non-zero components. However, we will carry it out for the sake of verification and mathematical interest. By introducing the imaginary part of (18) (19) and (20) in the expression of the magnetic energy

$$W = \int\int\int \frac{\mu}{2} H^2 r^2 \sin\theta dr d\theta d\varphi \tag{39}$$

and performing the integrals with respect to φ using (13) and (14), we obtain

$$W = \frac{\pi A^2}{2\omega^2\mu} M \int\int \{ \frac{\ell^2(\ell+1)^2}{r^3}\Psi^2_{\ell+1/2}(kr)[P^m_\ell(\cos\theta)]^2 + \frac{1}{r^2}\{\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\}^2 dr[\frac{d}{d\theta}P^m_l(\cos\theta)]^2 + \frac{m^2}{r^2\sin^2\theta} \{\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\}^2[P^m_l(\cos\theta)]^2\} r^2\sin\theta dr d\theta \tag{40}$$

or, using (11)

$$W = \frac{\epsilon\pi A^2}{2k^2} M \{\ell^2(\ell+1)^2 \int_a^R \frac{1}{r}\Psi^2_{\ell+1/2}(kr)dr \int_0^\pi \sin\theta[P^m_\ell(\cos\theta)]^2 d\theta + \int_0^R \{\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\}^2 dr \int_0^\pi \sin\theta[\frac{d}{d\theta}P^m_\ell(\cos\theta)]^2 d\theta + \int_a^R \{\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\}^2 dr \int_0^\pi \frac{m^2}{\sin\theta}[P^m_\ell(\cos\theta)]^2 d\theta\} \tag{41}$$

Performing the integrals with respect to θ , we obtain by (169) (167) and (168):

$$W = \frac{\epsilon\pi A^2}{2k^2} M \{\ell^2(\ell+1)^2 \int_a^R \frac{1}{r}\Psi^2_{\ell+1/2}(kr)dr \frac{1}{\ell+1/2} + \int_a^R \{\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\}^2 dr[\frac{\ell(\ell+1)}{\ell+1/2} - m] + \int_a^R \{\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\}^2 dr\ m\} \frac{(\ell+m)!}{(\ell-m)!} \tag{42}$$

or

$$W = \frac{\epsilon\pi A^2}{2k^2} M \frac{\ell(\ell+1)(\ell+m)!}{(\ell+1/2)(l-m)!} \int_a^R \{\ell(\ell+1)\frac{1}{r}\Psi^2_{\ell+1/2}(kr) + \{\frac{d}{dr}[sqrtr\Psi_{\ell+1/2}(kr)]\}^2\}dr \tag{43}$$

The integral in (43) is achieved using (166). We obtain

$$W = \frac{\epsilon\pi A^2}{2k^2} M \frac{\ell(\ell+1)(\ell+m)!}{(\ell+1/2)(\ell-m)!} \{[2kr\Psi'_{\ell+1/2}(kr) + \Psi_{\ell+1/2}(kr)] [\frac{kr}{4}\Psi'_{\ell+1/2}(kr) + \frac{3}{8}\Psi_{\ell+1/2}(kr)] + [\frac{k^2r^2}{2} - \frac{\ell(\ell+1)}{2}]\Psi^2_{\ell+1/2}(kr)\}|^R_a \tag{44}$$

Let using the boundary conditions (21) and (22)

$$W = \frac{\epsilon\pi A^2}{2k^2} M \frac{\ell(\ell+1)(\ell+m)!}{(\ell+1/2)(\ell-m)!} [\frac{k^2R^2}{2}\Psi'_{\ell+1/2}(kR) - \frac{k^2a^2}{2}\Psi'_{\ell+1/2}(ka)] \tag{45}$$

and finally

$$W = \frac{\epsilon\pi A^2}{4} M \frac{\ell(\ell+1)(\ell+m)!}{(\ell+1/2)(\ell-m)!} [R^2\Psi'^2_{\ell+1/2}(kR) - a^2\Psi'^2_{\ell+1/2}(ka)] \tag{46}$$

which is indeed identical to (38).

E. THRUST OF NORMAL MODES ON THE CAVITY WALLS

The thrust of normal modes on the walls at $r = a$ and $r = R$ is purely magnetic since the electric field (15) (16) (17) cancels out at these points by (21) and (22). Since the magnetic field is purely tangential (19) (20), the peak force density is, by virtue of Maxwell tensor:

$$\frac{1}{2}\mu(H_\theta^2 + H_\varphi^2) \tag{47}$$

As the magnetic field varies sinusoidally in time, a factor $\frac{1}{2}$ is introduced to account for the time average. The thrust, i.e., the integral of the force density, thus becomes, by introducing (19) and (20) in (47) divided by two and performing the integrals with respect to φ using (13) and (14):

$$\begin{aligned} F = {} & \frac{\pi A^2}{4\omega^2\mu} M \{\frac{1}{r}\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\}^2 \int \{\frac{d}{d\theta}[P_\ell^m(\cos\theta)]\}^2 r^2 \sin\theta d\theta \\ & + \frac{\pi A^2}{4\omega^2\mu} M \{\frac{1}{r}\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\}^2 \int \frac{m^2}{\sin^2\theta}\{[P_\ell^m(\cos\theta)]\}^2 r^2 \sin\theta d\theta \end{aligned} \tag{48}$$

By performing the derivatives with respect to r and using the boundary conditions (21) (22), as well as (11), the thrust is written

$$\begin{aligned} F = {} & \frac{\pi\epsilon A^2}{4k^2} M \{\frac{1}{r} k\sqrt{r}\Psi'_{\ell+1/2}(kr)\}^2 \int \{\frac{d}{d\theta}[P_\ell^m(\cos\theta)]\}^2 r^2 \sin\theta d\theta \\ & + \frac{\pi\epsilon A^2}{4k^2} M \{\frac{1}{r} k\sqrt{r}\Psi'_{\ell+1/2}(kr)\}^2 \int \frac{m^2}{\sin\theta}\{[P_\ell^m(\cos\theta)]\}^2 r^2 d\theta \end{aligned} \tag{49}$$

or

$$\begin{aligned} F = {} & \frac{\pi\epsilon A^2}{4} M r \Psi'^2_{\ell+1/2}(kr) \int_0^\pi \{\frac{d}{d\theta}[P_\ell^m(\cos\theta)]\}^2 \sin\theta d\theta \\ & + \frac{\pi\epsilon A^2}{4} M r \Psi'^2_{\ell+1/2}(kr) \int_0^\pi \frac{m^2}{\sin\theta}[P_\ell^m(\cos\theta)]^2 d\theta \end{aligned} \tag{50}$$

Performing the integrals with respect to θ , we obtain by (167) and (168)

$$F = \frac{\pi\epsilon A^2}{4} M r \Psi'^2_{\ell+1/2}(kr) \frac{\ell(\ell+1)}{\ell+1/2} \frac{(\ell+m)!}{(\ell-m)!} \tag{51}$$

By specifying (51) for $r = a$ and $r = R$, and then comparing the result with (38), we can write

$$F(a) = \frac{a\Psi'^2_{\ell+1/2}(ka)}{R^2\Psi'^2_{\ell+1/2}(kR) - a^2\Psi'^2_{\ell+1/2}(ka)} W \tag{52}$$

$$F(R) = \frac{R\Psi'^2_{\ell+1/2}(kR)}{R^2\Psi'^2_{\ell+1/2}(kR) - a^2\Psi'^2_{\ell+1/2}(ka)} W \tag{53}$$

Combining (52) and (53), we have the remarkable relationship

$$RF(R) - aF(a) = W \tag{54}$$

F. LOSSES DUE TO SURFACE CURRENTS

At the boundaries of the cavity, a surface current arises which, by Ampère's law, must be equal to the tangential component of the magnetic field. If the material that bounds the cavity is not perfectly conductive, but conductive enough for the losses to be small, the losses can be calculated by keeping the expressions (16) (17) for the field of the undamped normal modes. Surface currents encounter a surface resistance R_s which is

$$R_s = \frac{1}{\sigma\delta} \tag{55}$$

where σ is the conductivity of the metal and δ the skin depth:

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{56}$$

The peak power loss density is therefore given by

$$R_s(H_\theta^2 + H_\varphi^2) \tag{57}$$

The comparison of (57) and (47) shows that the calculations in the previous paragraph allow the losses to be calculated. It is sufficient to multiply the results (52) (53) by the fraction $2R_s/\mu$ That is, by (55), (56) and (9),

$$2\frac{R_s}{\mu} = \frac{2}{\sigma\mu\delta} = \sqrt{\frac{2\omega}{\sigma\mu}} = \sqrt{\frac{2kc}{\sigma\mu}} \tag{58}$$

whose value can be different in $r = a$ and in $r = R$ if the conductivity σ is not the same for both walls.

G. CASE OF SPHERICAL CAVITY

In the case of a spherical cavity, $a = 0$ and (52) is irrelevant. The continuity of the fields at the origin imposes

$$\Psi_{\ell+1/2}(kr) = J_{\ell+1/2}(kr) \tag{59}$$

Using the limit form [6](9.1.7), it is easy to check that

$$\lim_{a\to 0} a\Psi'^2_{\ell+1/2}(ka) = 0 \tag{60}$$

By introducing (60) into (54), we see that the thrust (53) reduces to

$$F(R) = \frac{W}{R} \quad if \quad a = 0 \tag{61}$$

H. EXTERIOR MODES OF SPHERES

Furthermore, if we study the exterior of spheres of radius a when $R \to \infty$, the asymptotic form (3) introduced in the boundary condition (22) shows that we must have

$$\cos(kR - \frac{1}{2}\ell\pi - \frac{1}{2}\pi + \zeta) = 0 \qquad if \qquad R\, is\, large \quad (62)$$

and therefore also

$$\sin(kR - \frac{1}{2}\ell\pi - \frac{1}{2}\pi + \zeta) = \pm 1 \qquad if \qquad R\, is\, large \quad (63)$$

Using (3), (62) and (63), we obtain

$$\Psi'_{\ell+1/2}(kR) \approx \pm\sqrt{\frac{2}{\pi kR}} \qquad if \qquad R\, is\, large \quad (64)$$

It can be deduced that

$$\lim_{R\to\infty} R\Psi'^2_{\ell+1/2}(kR) = \frac{2}{\pi k} \quad (65)$$

Introducing (65) into (52) and (53), we obtain

$$F(a) \approx \frac{\pi ka \Psi'^2_{\ell+1/2}(ka)}{2R} W \qquad if \qquad R \to \infty \quad (66)$$

$$F(R) \approx \frac{W}{R} \qquad if \qquad R \to \infty \quad (67)$$

Using (31), equation (66) can be written as

$$F(a) \approx \frac{2}{\pi ka} \frac{1}{J^2_{\ell+1/2}(ka) + Y^2_{\ell+1/2}(ka)} \frac{W}{R} \qquad if\ R \to \infty \quad (68)$$

III. STUDY OF MODES TM

A. EXPRESSION OF THE FIELDS

Apart from parity, the wave fields $TM_{\ell mn}$ are of the form given in [2], i.e., after generalization:

$$E_r = \frac{\ell(\ell+1)B}{j\omega\epsilon r^{3/2}} \Psi_{\ell+1/2}(kr) P_\ell^m(\cos\theta)\cos(m\varphi) \quad (69)$$

$$E_\theta = \frac{B}{j\omega\epsilon r}\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\frac{d}{d\theta}[P_\ell^m(\cos\theta)]\cos(m\varphi) \quad (70)$$

$$E_\varphi = \frac{mB}{j\omega\epsilon r\sin\theta}\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]P_\ell^m(\cos\theta)\sin(m\varphi) \quad (71)$$

and

$$H_r = 0 \quad (72)$$

$$H_\theta = \frac{mB}{\sqrt{r}\sin\theta}\Psi_{\ell+1/2}(kr)P_\ell^m(\cos(\theta))\sin(m\varphi) \quad (73)$$

$$H_\varphi = \frac{B}{\sqrt{r}}\Psi_{\ell+1/2}(kr)\frac{d}{d\theta}[P_\ell^m(\cos\theta)]\cos(m\varphi) \quad (74)$$

B. BOUNDARY CONDITIONS

Since the walls of the cavity are perfectly conductive, in $r = a$ and $r = R$ the tangential component of the electric field and the radial component of the magnetic field must be zero. It is immediately clear that these conditions are satisfied if

$$\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)] = 0 \qquad for\ r = a\ and\ for\ r = R. \quad (75)$$

In order to shorten the writings, let us define for any function $F(x)$: $\bar{F}(x) = F(x) + 2xF'(x)$. Then, by performing the derivation of (75), we obtain the conditions:

$$\Psi_{\ell+1/2}(ka) + 2ka\Psi'_{\ell+1/2}(ka) = \bar{\Psi}_{\ell+1/2}(ka) = 0 \quad (76)$$

and

$$\Psi_{\ell+1/2}(kR) + 2kR\Psi'_{\ell+1/2}(kR) = \bar{\Psi}_{\ell+1/2}(kR) = 0 \quad (77)$$

Considering (1), equations (76) and (77) can be written:

$$\tan(\zeta) = \frac{\bar{J}_{\ell+1/2}(ka)}{\bar{Y}_{\ell+1/2}(ka)} \quad (78)$$

and

$$\tan(\zeta) = \frac{\bar{J}_{\ell+1/2}(kR)}{\bar{Y}_{\ell+1/2}(kR)} \quad (79)$$

Identifying (78) and (79) gives an equation defining the values of k. Thus, the two conditions (76) (77) are satisfied for a discrete set of pairs (k, ζ). From (78) and (79), we also obtain, with $\pm 1 = sgn[\bar{J}_{\ell+1/2}(ka)]$

$$\cos(\zeta) = \frac{\pm\bar{Y}_{\ell+1/2}(ka)}{\sqrt{\bar{J}^2_{\ell+1/2}(ka) + \bar{Y}^2_{\ell+1/2}(ka)}} \quad (80)$$

$$\sin(\zeta) = \frac{\pm\bar{J}_{\ell+1/2}(ka)}{\sqrt{\bar{J}^2_{\ell+1/2}(ka) + \bar{Y}^2_{\ell+1/2}(ka)}} \quad (81)$$

And, with $\pm 1 = sgn[\bar{J}_{\ell+1/2}(kR)]$

$$\cos(\zeta) = \frac{\pm\bar{Y}_{\ell+1/2}(kR)}{\sqrt{\bar{J}^2_{\ell+1/2}(kR) + \bar{Y}^2_{\ell+1/2}(kR)}} \quad (82)$$

$$\sin(\zeta) = \frac{\pm\bar{J}_{\ell+1/2}(kR)}{\sqrt{\bar{J}^2_{\ell+1/2}(kR) + \bar{Y}_{\ell+1/2}(kR)}} \quad (83)$$

We can obtain an interesting expression of $\Psi_{\ell+1/2}(ka)$. Using (1), we have

$$\Psi_{\ell+1/2}(ka) = J_{\ell+1/2}(ka)\cos(\zeta) - Y_{\ell+1/2}(ka)\sin(\zeta) \quad (84)$$

Introducing (80) and (81) in (84), it comes

$$\Psi_{\ell+1/2}(ka) = \pm\frac{J_{\ell+1/2}(ka)[\bar{Y}_{\ell+1/2}(ka)] - Y_{\ell+1/2}(ka)[\bar{J}_{\ell+1/2}(ka)]}{\sqrt{\bar{J}^2_{\ell+1/2}(ka) + \bar{Y}^2_{\ell+1/2}(ka)}} \quad (85)$$

or

$$\Psi_{\ell+1/2}(ka) = \pm 2ka \frac{J_{\ell+1/2}(ka)Y'_{\ell+1/2}(ka) - Y_{\ell+1/2}(ka)J'_{\ell+1/2}(ka)}{\sqrt{\bar{J}^2_{\ell+1/2}(ka) + \bar{Y}^2_{\ell+1/2}(ka)}} \quad (86)$$

The numerator of (86) is a Wronskian. Using its expression [6](9.1.16)(9.1.27), we obtain

$$\Psi_{\ell+1/2}(ka) = \pm \frac{4}{\pi} \frac{1}{\sqrt{\bar{J}^2_{\ell+1/2}(ka) + \bar{Y}^2_{\ell+1/2}(ka)}} \quad (87)$$

Of course, one has the similar expression for $r = R$:

$$\Psi_{\ell+1/2}(kR) = \pm \frac{4}{\pi} \frac{1}{\sqrt{\bar{J}^2_{\ell+1/2}(kR) + \bar{Y}^2_{\ell+1/2}(kR)}} \quad (88)$$

C. ENERGY CALCULATION VIA THE MAGNETIC FIELD

The simplest way to calculate the energy of a TM normal mode is to do the calculation at the moment when the electric field is zero, since at that moment all the energy is in magnetic form and the magnetic field (72) (73) (74) has only two non-zero component. By introducing these components into the expression for the magnetic energy (39), i.e.

$$W = \int\int\int \frac{\mu}{2} H^2 r^2 \sin\theta dr d\theta d\varphi \quad (89)$$

and performing the integrals with respect to φ using (13) and (14), we obtain

$$W = \frac{\mu\pi B^2}{2} M \int\int \{ \frac{m^2}{r \sin^2\theta} \Psi^2_{\ell+1/2}(kr)[P^m_\ell(\cos\theta)]^2 + \frac{1}{r}\Psi^2_{\ell+1/2}(kr)[\frac{d}{d\theta}P^m_\ell(\cos\theta)]^2\} r^2 \sin\theta dr d\theta \quad (90)$$

or

$$W = \frac{\mu\pi B^2}{2} M [\int_a^R r\Psi^2_{\ell+1/2}(kr)dr] \int_0^\pi \{\frac{m^2}{\sin\theta}[P^m_\ell(\cos\theta)]^2 + \sin\theta[\frac{d}{d\theta}P^m_\ell(\cos\theta)]^2\} d\theta \quad (91)$$

The integral with respect to θ is obtained using (168) and (167).

$$W = \frac{\mu\pi B^2}{2} M \frac{(\ell+m)!}{(\ell-m)!} [\int_a^R r\Psi^2_{\ell+1/2}(kr)dr]\{m + \frac{\ell(\ell+1)}{\ell+1/2} - m\} \quad (92)$$

$$W = \frac{\mu\pi B^2}{2} M [\int_a^R r\Psi^2_{\ell+1/2}(kr)dr] \frac{\ell(\ell+1)}{\ell+1/2} \frac{(\ell+m)!}{(\ell-m)!} \quad (93)$$

The integral with respect to r is a Lommel integral. It is performed using (165), We obtain

$$W = \frac{\mu\pi B^2}{2} M \frac{\ell(\ell+1)}{\ell+1/2} \frac{(\ell+m)!}{(\ell-m)!} \{\frac{R^2}{2}[\Psi'^2_{\ell+1/2}(kR) + (1 - \frac{(\ell+1/2)^2}{k^2R^2})\Psi^2_{\ell+1/2}(kR)] - \frac{a^2}{2}[\Psi'^2_{\ell+1/2}(ka) + (1 - \frac{(\ell+1/2)^2}{k^2a^2}\Psi^2_{\ell+1/2}(ka)]\} \quad (94)$$

The derivatives can be eliminated by the boundary conditions (76) (77), and we obtain

$$W = \frac{\mu\pi B^2}{2} M \frac{\ell(\ell+1)}{\ell+1/2} \frac{(\ell+m)!}{(\ell-m)!} [\frac{R^2}{2}(1 - \frac{(\ell+1/2)^2}{k^2R^2} - \frac{1}{4k^2R^2})\Psi^2_{\ell+1/2}(kR) - \frac{a^2}{2}(1 - \frac{(\ell+1/2)^2}{k^2a^2} + \frac{1}{4k^2a^2})\Psi^2_{\ell+1/2}(ka)] \quad (95)$$

or

$$W = \frac{\mu\pi B^2}{4} M \frac{\ell(\ell+1)}{\ell+1/2} \frac{(\ell+m)!}{(\ell-m)!} [R^2(1 - \frac{\ell(\ell+1)}{k^2R^2})\Psi^2_{\ell+1/2}(kR) - a^2(1 - \frac{\ell(\ell+1)}{k^2a^2})\Psi^2_{\ell+1/2}(ka)] \quad (96)$$

D. CALCULATING OF ENERGY VIA THE ELECTRIC FIELD

The energy expression can also be obtained by integrating the electrical energy taken at the moment where the magnetic field is zero. The calculation is more difficult than in the previous paragraph because the electric field has three non-zero components. However, we will do it for the sake of overlap. By introducing the imaginary part of the electric field expression (69) (70) (71) into the electric expression (33), i.e.

$$W = \int\int\int \frac{\epsilon}{2} E^2 r^2 \sin\theta dr d\theta d\varphi \quad (97)$$

and performing the integrals with respect to φ using (13) and (14), we obtain

$$W = \frac{\pi B^2}{2\omega^2\epsilon} M \int\int \{\frac{\ell^2(\ell+1)^2}{r}\Psi^2_{\ell+1/2}(kr)[P^m_\ell(\cos\theta)]^2 + \{\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\}^2[\frac{d}{d\theta}P^m_\ell(\cos\theta)]^2 + \frac{m^2}{\sin^2\theta}\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\}^2[P^m_\ell(\cos\theta)]^2\} \sin\theta dr d\theta \quad (98)$$

or, using (11) and performing the integrals with respect to θ by (169) (167) and (168)

$$W=\frac{\mu\pi B^2}{2k^2}M\{\ell^2(\ell+1)^2\int_a^R\frac{1}{r}\Psi^2_{\ell+1/2}(kr)dr\frac{1}{\ell+1/2}+\int_a^R\{\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\}^2dr[\frac{\ell(\ell+1)}{\ell+1/2}-m]+\int_a^R\{\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\}^2dr\ m\}\frac{(\ell+m)!}{(\ell-m)!} \tag{99}$$

That is, if we combine the last two integrals

$$W=\frac{\mu\pi B^2}{2k^2}M\frac{\ell(\ell+1)(\ell+m)!}{(\ell+1/2)(\ell+m)!}\int_a^R\{\ell(\ell+1)\frac{1}{r}\Psi^2_{\ell+1/2}(kr)dr+\{\frac{d}{dr}[\sqrt{r}\Psi_{\ell+1/2}(kr)]\}^2\}dr \tag{100}$$

The integral in (100) is solved using (166). We obtain

$$W=\frac{\mu\pi B^2}{2k^2}M\frac{\ell(\ell+1)(\ell+m)!}{(\ell+1/2)(\ell+m)!}\{\bar\Psi_{\ell+1/2}(kr)[\frac{kr}{4}\Psi'_{\ell+1/2}(kr)+\frac{3}{8}\Psi_{\ell+1/2}(kr)]+[\frac{k^2r^2}{2}-\frac{\ell(\ell+1}{2}]\Psi^2_{\ell+1/2}(kr)\}|_a^R \tag{101}$$

Let, using the boundary conditions (76) and (77)

$$W=\frac{\mu\pi B^2}{2k^2}M\frac{\ell(\ell+1)}{\ell+1/2}\frac{(\ell+m)!}{(\ell+m)!}[(\frac{k^2R^2}{2}-\frac{\ell(\ell+1}{2})\Psi^2_{\ell+1/2}(kR)-(\frac{k^2a^2}{2}-\frac{\ell(\ell+1)}{2})\Psi^2_{\ell+1/2}(ka)] \tag{102}$$

and finally

$$W=\frac{\mu\pi B^2}{4}M\frac{\ell(\ell+1)}{\ell+1/2}\frac{(\ell+m)!}{(\ell+m)!}[R^2(1-\frac{\ell(\ell+1}{k^2R^2})\Psi^2_{\ell+1/2}(kR)-a^2(1-\frac{\ell(\ell+1}{k^2a^2})\Psi^2_{\ell+1/2}(ka)] \tag{103}$$

which is indeed identical to (96).

E. CALCULATION OF THE MAGNETIC THRUST

In the case of TM modes, there is at the boundaries both a tangential magnetic field component and a normal electric field component. The thrust due to the magnetic field is directed outwards as the thrust in the case of TE modes. In contrast, the electric field gives rise to an inward pull. The resulting thrust can therefore be written as

$$F=F_\mu-F_\epsilon \tag{104}$$

We will calculate the two right-hand terms separately because only the calculation of F_μ can be reused for the calculation of losses. Since the magnetic field is purely tangential, the force density is, by virtue of Maxwell's tensor, given by (47), i.e.:

$$\frac{1}{2}\mu(H^2_\theta+H^2_\varphi) \tag{105}$$

As the magnetic field varies sinusoidally in time, a factor $1/2$ must be introduced to account for the time average. The integral of the force density (105) divided by two becomes, introducing (73) et (74) and performing the integrals according to φ using (13) and (14):

$$F_\mu=\frac{\mu\pi B^2}{4}M\frac{1}{r}\Psi^2_{\ell+1/2}(kr)\{\int\frac{m^2}{\sin^2\theta}[P^m_\ell(\cos\theta)]^2+\{\frac{d}{d\theta}[P^m_\ell(\cos\theta)]\}^2\}r^2\sin\theta d\theta \tag{106}$$

The integral with respect to θ is obtained using (167) and (168).

$$F_\mu=\frac{\mu\pi B^2}{4}Mr\Psi^2_{\ell+1/2}(kr)(m+\frac{\ell(\ell+1)}{\ell+1/2}-m)\frac{(\ell+m)!}{(\ell-m)!} \tag{107}$$

so

$$F_\mu(a)=\frac{\mu\pi B^2}{4}Ma\Psi^2_{\ell+1/2}(ka)\frac{\ell(\ell+1)}{(\ell+1/2)}\frac{(\ell+m)!}{(\ell-m)!} \tag{108}$$

$$F_\mu(R)=\frac{\mu\pi B^2}{4}MR\Psi^2_{\ell+1/2}(kR)\frac{\ell(\ell+1)}{(\ell+1/2)}\frac{(\ell+m)!}{(\ell-m)!} \tag{109}$$

F. LOSSES CALCULATION

As in the case of the TE modes, to obtain the expression for the losses it is sufficient to multiply the expression for the magnetic thrusts (108) or (109) as the case may be, by the factor (58), i.e.:

$$2\frac{R_s}{\mu}=\frac{2}{\sigma\mu\delta}=\sqrt{\frac{2\omega}{\sigma\mu}}=\sqrt{\frac{2kc}{\sigma\mu}} \tag{110}$$

G. CALCULATION OF THE ELECTRIC PULL AND THE RESULTING THRUST

In the case of TM mode, the radial component of the electric field is not zero at the boundaries. Therefore, the Maxwell tensor gives a force density facing the interior of the cavity, which is

$$\frac{1}{2}\epsilon E^2_r \tag{111}$$

As the electric field varies sinusoidally in time, a factor $1/2$ is introduced to account for the time average. Introducing (69) in (111) divided by two and performing the integral with respect to φ by (13), we obtain

$$F_e(r)=\frac{\pi}{4}\frac{\ell^2(\ell+1)^2B^2}{\omega^2\varepsilon}M\frac{1}{r}\Psi^2_{\ell+1/2}(kr)\int[P^m_\ell(\cos\theta)]^2\sin(\theta)d\theta \tag{112}$$

The integral with respect to θ is performed by (169). Using (11), we obtain

$$F_e(r) = \frac{\pi}{4}\frac{\ell^2(\ell+1)^2\mu B^2}{k^2}M\frac{1}{r}\Psi^2_{\ell+1/2}(kr)\frac{(\ell+m)!}{(\ell+1/2)(\ell-m)!} \quad (113)$$

thus

$$F_e(a) = \frac{\pi}{4}\frac{\mu B^2}{k^2}M\frac{1}{r}\Psi^2_{\ell+1/2}(ka)\frac{\ell^2(\ell+1)^2}{(\ell+1/2)}\frac{(\ell+m)!}{(\ell-m)!} \quad (114)$$

and

$$F_e(R) = \frac{\pi}{4}\frac{\mu B^2}{k^2}M\frac{1}{r}\Psi^2_{\ell+1/2}(kR)\frac{\ell^2(\ell+1)^2}{(\ell+1/2)}\frac{(\ell+m)!}{(\ell-m)!} \quad (115)$$

To find the expression for the resultant thrust, it is sufficient to introduce (108) and (114) or (109) and (115) into (104). This gives us,

$$F(a) = \frac{\mu\pi B^2}{4}Ma(1-\frac{\ell(\ell+1)}{k^2a^2})\Psi^2_{\ell+1/2}(ka)\frac{\ell(\ell+1)}{(\ell+1/2)}\frac{(\ell+m)!}{(\ell-m)!} \quad (116)$$

$$F(R) = \frac{\mu\pi B^2}{4}Ma(1-\frac{\ell(\ell+1)}{k^2a^2})\Psi^2_{\ell+1/2}(kR)\frac{\ell(\ell+1)}{(\ell+1/2)}\frac{(\ell+m)!}{(\ell-m)!} \quad (117)$$

Comparing the expressions (116) or (117) with those of energy (96), we see that we have:

$$F(a) = \frac{a(1-\frac{\ell(\ell+1)}{k^2a^2})\Psi^2_{\ell+1/2}(ka)}{denoma}W \quad with$$
$$denoma = R^2(1-\frac{\ell(\ell+1)}{k^2R^2})\Psi^2_{\ell+1/2}(kR) - a^2(1-\frac{\ell(\ell+1)}{k^2a^2})\Psi^2_{\ell+1/2}(ka) \quad (118)$$

$$F(R) = \frac{R(1-\frac{\ell(\ell+1)}{k^2a^2})\Psi^2_{\ell+1/2}(kR)}{denomR}W \quad with$$
$$denomR = R^2(1-\frac{\ell(\ell+1)}{k^2R^2})\Psi^2_{\ell+1/2}(kR) - a^2(1-\frac{\ell(\ell+1)}{k^2a^2})\Psi^2_{\ell+1/2}(ka) \quad (119)$$

Combining (118) and (119), we obtain a relationship identical to the one that was found for the TE modes, namely

$$RF(R) - aF(a) = W \quad (120)$$

H. CASE OF SPHERICAL CAVITY

In the case of a spherical cavity ($a \to 0$), (119) provides the same relationship (61) then in the case of TE, i.e.:

$$F(R) = \frac{W}{R} \quad (121)$$

I. EXTERIOR MODES OF SPHERES

Furthermore, if we study the exterior of spheres of radius a when $R \to \infty$, the limit form (3) gives

$$\Psi'_{\ell+1/2}(kR) \approx \sqrt{\frac{2}{\pi}}(-\frac{1}{2})\frac{1}{\sqrt{(kR)^3}}\cos(kR-\frac{1}{2}\ell\pi-\frac{1}{2}\pi+\zeta) - \sqrt{\frac{2}{\pi kR}}\sin(kR-\frac{1}{2}\ell\pi-\frac{1}{2}\pi+\zeta) \quad (122)$$

Introducing (122) into the boundary condition (77), we obtain

$$\sqrt{\frac{2}{\pi kR}}\sin(kR-\frac{1}{2}\ell\pi-\frac{1}{2}\pi+\zeta) - 2kR\sqrt{\frac{2}{\pi}}(-\frac{1}{2})\frac{1}{\sqrt{(kR)^3}}\cos(kR-\frac{1}{2}\ell\pi-\frac{1}{2}\pi+\zeta) + kR\sqrt{\frac{2}{\pi kR}}\sin(kR-\frac{1}{2}\ell\pi-\frac{1}{2}\pi+\zeta) \approx 0 \quad (123)$$

or

$$\sqrt{\frac{2}{\pi kR}}\sin(kR-\frac{1}{2}\ell\pi-\frac{1}{2}\pi+\zeta) \approx 0 \quad (124)$$

so

$$\sin(kR-\frac{1}{2}\ell\pi-\frac{1}{2}\pi+\zeta) \approx 0 \quad if\ R\ is\ large \quad (125)$$

and therefore also

$$\cos(kR-\frac{1}{2}\ell\pi-\frac{1}{2}\pi+\zeta) \approx \pm 1 \quad if\ R\ is\ large \quad (126)$$

Using (3) again, given (125) and (126)

$$\Psi_{\ell+1/2}(kR) \approx \pm\sqrt{\frac{2}{\pi kR}} \quad (127)$$

It can be deduced from (127) that

$$\lim_{R\to\infty} R\Psi^2_{\ell+1/2}(kR) = \frac{2}{\pi k} \quad (128)$$

By introducing (128) into (118) and (119), we obtain

$$F(a) \approx \frac{\pi ka(1-\frac{\ell(\ell+1)}{k^2a^2})\Psi^2_{\ell+1/2}(ka)}{2R}W \quad if\ R\to\infty \quad (129)$$

$$F(R) \approx \frac{1}{R}W \quad if\ R\to\infty \quad (130)$$

Using (87), the expression (129) can be written

$$F(a) \approx \frac{8ka}{\pi}\frac{1-\frac{\ell(\ell+1)}{k^2a^2}}{\bar{J}^2_{\ell+1/2}(ka)+\bar{Y}^2_{\ell+1/2}(ka)}\frac{W}{R} \quad if\ R\to\infty \quad (131)$$

Although (130) is identical to (67), the difference between (129) and (66), or between (131) and (67), should be noted.

IV. CONCLUSIONS

We have obtained analytical formulae expressing the energy and losses of all modes of a spherical annular cavity. These formulae apply to the case of spherical cavities by making the inner radius tend toward 0. We also obtained simple relationships between the thrust exerted by the modes and their energy, as well as the relationships between the surface losses of the modes and their energy.

V. EPILOGUE

It is known that each normal mode has at low temperature, an energy

$$W = \frac{1}{2}\hbar\omega \tag{132}$$

where $\hbar$ is Plank's constant divided by 2π. Given (9), we have

$$W = \frac{1}{2}\hbar ck \tag{133}$$

The spherical Casimir effect was calculated by Boyer [3] by assuming that the thrust is the derivative of the energy calculated by (133) with respect to the radius, i.e.

$$F(a) = \frac{dW}{da} \tag{134}$$

and

$$F(R) = -\frac{dW}{dR} \tag{135}$$

We consider this to be a conjecture because, since the system is not closed, conservation of energy cannot be invoked to justify (134) and (135).

We assume that if relations (134) and (135) are valid for the total of the modes, they must also be valid for each mode separately. We should therefore have

$$F(a) = \frac{1}{2}\hbar c\frac{dk}{da} \tag{136}$$

and

$$F(R) = -\frac{1}{2}\hbar c\frac{dk}{dR} \tag{137}$$

A. CASE OF SPHERICAL CAVITY

In the interior case of a spherical cavity, conditions (22) and (77) lead, given (2), to a relationship between k and R of the form

$$kR = \kappa \tag{138}$$

where κ is a different constant for each mode.

From (138), we derive

$$dk = -\frac{\kappa}{R^2}dR \tag{139}$$

Introducing (139) into (137) and again using (133) and (138), we obtain

$$F(R) = -\frac{1}{2}\hbar c(-\frac{\kappa}{R^2}) = \frac{1}{2}\hbar c\frac{\kappa}{R}\frac{1}{R} = \frac{1}{2}\hbar c\frac{\kappa}{R^2} \tag{140}$$

which is consistent with the Casimir's hypothesis [3], and

$$F(R) = \frac{1}{2}\hbar ck\frac{1}{R} = \frac{W}{R} \tag{141}$$

which is consistent with (67) and (121).

B. EXTERIOR MODES OF SPHERES (GENERALITIES)

However, in the case of a spherical annular cavity, the situation is different because the coefficient ζ varies with k and therefore with a . We limit our analysis to the case where R is very large. In this case, by (62) or (125), we have

$$d\zeta = -Rdk \tag{142}$$

C. EXTERIOR TE MODES OF SPHERES

Let us first consider the TE modes. Condition (21) provides

$$a\Psi'_{\ell+1/2}(ka)dk + k\Psi'_{\ell+1/2}(ka)da + [\partial\Psi_{\ell+1/2}(ka)/\partial\zeta]d\zeta = 0 \tag{143}$$

Therefore, using (142),

$$\{a\Psi'_{\ell+1/2}(ka) - R[\partial\Psi_{\ell+1/2}(ka)/\partial\zeta]\}dk = -k\Psi'_{\ell+1/2}(ka)da \tag{144}$$

$$\frac{dk}{da} = -\frac{k\Psi'_{\ell+1/2}(ka)}{a\Psi'_{\ell+1/2}(ka) - R[\partial\Psi_{\ell+1/2}(ka)/\partial\zeta]} \tag{145}$$

and so, since R is large,

$$\frac{dk}{da} = \frac{k\Psi'_{\ell+1/2}(ka)}{R[\partial\Psi_{\ell+1/2}(ka)/\partial\zeta]} \tag{146}$$

By (136), we obtain

$$F(a) = \frac{1}{2}\hbar c\frac{dk}{da} = \frac{1}{2}\hbar c\frac{k\Psi'_{\ell+1/2}(ka)}{R[\partial\Psi_{\ell+1/2}(ka)/\partial\zeta]} = \frac{W}{R}\frac{\Psi'_{\ell+1/2}(ka)}{\partial\Psi_{\ell+1/2}(ka)/\partial\zeta} \tag{147}$$

Using (1), (147) becomes

$$F(a) = -\frac{W}{R}\frac{J'_{\ell+1/2}(ka)\cos(\zeta) - Y'_{\ell+1/2}(ka)\sin(\zeta)}{J_{\ell+1/2}(ka)\sin(\zeta) + Y_{\ell+1/2}(ka)\cos(\zeta)} \tag{148}$$

Introducing (25) and (26) in (148) and simplifying, one obtains

$$F(a) = \frac{W}{R}\frac{J'_{\ell+1/2}(ka)Y_{\ell+1/2}(ka) - Y'_{\ell+1/2}(ka)J_{\ell+1/2}(ka)}{J^2_{\ell+1/2}(ka) + Y^2_{\ell+1/2}(ka)} \tag{149}$$

The numerator of (149) is the Wronskian of $J_{\ell+1/2}(ka)$ and $Y_{\ell+1/2}(ka)$. Its value is given in [6](9.1.16). So, (149) becomes

$$F(a) = \frac{W}{R}\frac{2}{\pi ka}\frac{1}{J^2_{\ell+1/2}(ka) + Y^2_{\ell+1/2}(ka)} \tag{150}$$

which is equal to (68).

D. EXTERIOR TM MODES OF SPHERES

Consider now the TM modes. Condition (76) provides

$$\begin{aligned} a[\Psi'_{\ell+1/2}(ka) + 2ka\Psi"_{\ell+1/2}(ka) + 2\Psi'_{\ell+1/2}(ka)]dk \\ +k[\Psi'_{\ell+1/2}(ka) + 2ka\Psi"_{\ell+1/2}(ka) + 2\Psi'_{\ell+1/2}(ka)]da \\ -[\partial\Psi_{\ell+1/2}(ka)/\partial\zeta + 2ka\partial\Psi'_{\ell+1/2}(ka)/\partial\zeta]d\zeta = 0 \end{aligned} \quad (151)$$

or, using again (142), we obtain since R is large

$$\frac{dk}{da} = \frac{k}{R}\frac{3\Psi'_{\ell+1/2}(ka) + 2ka\Psi"_{\ell+1/2}(ka)}{\partial\Psi_{\ell+1/2}(ka)/\partial\zeta + 2ka\partial\Psi'_{\ell+1/2}(ka)/\partial\zeta} \quad (152)$$

Thus, introducing (152) into (136), we obtain using (133):

$$F(a) = \frac{W}{R}\frac{3\Psi'_{\ell+1/2}(ka) + 2ka\Psi"_{\ell+1/2}(ka)}{\partial\Psi_{\ell+1/2}(ka)/\partial\zeta + 2ka\partial\Psi'_{\ell+1/2}(ka)/\partial\zeta} \quad (153)$$

From the differential equation [6](9.1.1) which define the Bessel's functions, we have

$$\begin{aligned} ka\Psi"_{\ell+1/2}(ka) + \Psi'_{\ell+1/2}(ka) \\ = -\frac{1}{ka}[k^2a^2 - (\ell+1/2)^2]\Psi_{\ell+1/2}(ka) \end{aligned} \quad (154)$$

Thus, (153) becomes

$$F(a) = -\frac{W}{R}\frac{\Psi'_{\ell+1/2}(ka) - \frac{2}{ka}[k^2a^2 - (\ell+1/2)^2]\Psi_{\ell+1/2}}{\partial\Psi_{\ell+1/2}(ka)/\partial\zeta + 2ka\partial\Psi'_{\ell+1/2}(ka)/\partial\zeta} \quad (155)$$

Using (1), (155) becomes

$$F(a) = -\frac{W}{R}\frac{Num1}{[\bar{J}_{\ell+1/2}(ka)]\sin\zeta + [\bar{Y}_{\ell+1/2}(ka)]\cos\zeta} \quad (156)$$

with

$$\begin{aligned} Num1 = \\ [J'_{\ell+1/2}(ka) - 2ka[k^2a^2 - (\ell+1/2)^2]J_{\ell+1/2}(ka)]\cos\zeta \\ -[Y'_{\ell+1/2}(ka) - 2ka[k^2a^2 - (\ell+1/2)^2]Y_{\ell+1/2}(ka)]\sin\zeta \end{aligned} \quad (157)$$

Introducing (80) and (81) in (156,157), we obtain

$$F(a) = \frac{W}{R}\frac{Num2}{\bar{J}^2_{\ell+1/2}(ka) + \bar{Y}^2_{\ell+1/2}(ka)} \quad (158)$$

with

$$\begin{aligned} Num2 = [J'_{\ell+1/2}(ka) - 2[ka - \frac{(\ell+1/2)^2}{ka}]J_{\ell+1/2}(ka)] \\ [Y_{\ell+1/2}(ka) + 2kaY'_{\ell+1/2}(ka)] \\ -[Y'_{\ell+1/2}(ka) - 2[ka - \frac{(\ell+1/2)^2}{ka}]Y_{\ell+1/2}(ka)] \\ [J_{\ell+1/2} + 2kaJ'_{\ell+1/2}(ka)] \end{aligned} \quad (159)$$

or

$$\begin{aligned} Num2 = [J'_{\ell+1/2}(ka)Y_{\ell+1/2}(ka) - Y'_{\ell+1/2}(ka)J_{\ell+1/2}(ka)] \\ -4[k^2a^2 - (\ell+1/2)^2] \\ [J_{\ell+1/2}(ka)Y'_{\ell+1/2}(ka) - Y_{\ell+1/2}(ka)J'_{\ell+1/2}(ka)] \end{aligned} \quad (160)$$

thus

$$\begin{aligned} Num2 = [4k^2a^2 - 4(\ell+1/2)^2 + 1] \\ [J_{\ell+1/2}(ka)Y'_{\ell+1/2}(ka) - Y_{\ell+1/2}(ka)J'_{\ell+1/2}(ka)] \end{aligned} \quad (161)$$

and finally, replacing the Wronskian of (161) by its value [6](9.1.16) and inserting the result in (158)

$$F(a) = \frac{W}{R}\frac{8}{\pi ka}\frac{k^2a^2 - \ell(\ell+1)}{\bar{J}^2_{\ell+1/2}(ka) + \bar{Y}^2_{\ell+1/2}(ka)} \quad (162)$$

or

$$F(a) = \frac{W}{R}\frac{8ka}{\pi}\frac{1 - \frac{\ell(\ell+1)}{k^2a^2}}{\bar{J}^2_{\ell+1/2}(ka) + \bar{Y}^2_{\ell+1/2}(ka)} \quad (163)$$

which is identical to (131).

E. CONCLUSION

The expressions of the thrust (61), (68), (121) and (131) obtained by Maxwell theory are respectively identical to the expressions (141), (150), (141) and (163) obtained using for each normal mode the Boyer conjecture. In conclusion, Boyer's conjecture does fully apply to each eigenmode separately, at least for the limit cases implicated in spherical Casimir effect.

F. REMARK

When $R \to \infty$, the thrust (150) or (163) of one mode tends to 0 . However, the number of modes becomes very dense. Equation (62) or (125) shows that the gap between two consecutive modes is only

$$\Delta k = \pi/R \quad (164)$$

Multiplying (150) or (163) by the limit of the mode density R/π , inverse of (164), one sees that the limit of a sum of modes, i.e. $\int F(a)\frac{R}{\pi}dk$ is nonzero.

APPENDIX: SOME INTEGRALS

We use formulas demonstrated in the preliminary work [4]

$$\int x\Psi^2_\nu(\alpha x)dx = \frac{x^2}{2}[\Psi'^2_\nu(\alpha x) + (1 - \frac{\nu^2}{\alpha^2x^2})\Psi^2_\nu(\alpha x)] + cst \quad (165)$$

$$\begin{aligned} \int\{(\nu^2 - 1/4)\frac{1}{r}\Psi^2_\nu(kr) + \frac{d}{dr}[\sqrt{r}\Psi_\nu(kr)]^2\}dr \\ = [2kr\Psi'_\nu(kr) + \Psi_\nu(kr)][\frac{kr}{4}\Psi'_\nu(kr) + \frac{3}{8}\Psi_\nu(kr)] \\ +[\frac{k^2r^2}{2} - \frac{\nu^2 - 1/4}{2}]\Psi^2_\nu(kr) + cst \end{aligned} \quad (166)$$

$$\int_0^\pi \sin\theta[\frac{d}{d\theta}P^m_\ell(\cos\theta)]^2d\theta = [\frac{\ell(\ell+1)}{\ell+1/2} - m]\frac{(\ell+m)!}{(\ell-m)!} \quad (167)$$

$$\int_0^\pi \frac{m^2}{\sin\theta}[P^m_\ell(\cos\theta)]^2d\theta = \frac{m(\ell+m)!}{(\ell-m)!} \quad (168)$$

From [6](8.14.13), it is easy to obtain also

$$\int_0^\pi \sin\theta[P^m_\ell(\cos\theta)]^2d\theta = \frac{(\ell+m)!}{(\ell+1/2)(\ell-m)!} \quad (169)$$

Acknowledgements : Thanks to Alexandre Matagne for providing language help.

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